Building the Mandelbrot Set with Orbits

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MSS Maths Talks

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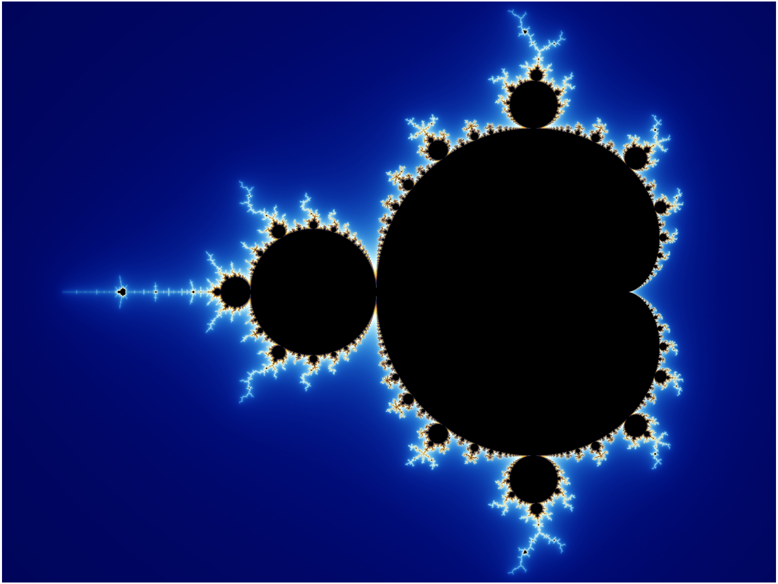
Overview

A Preface - Pretty Pictures



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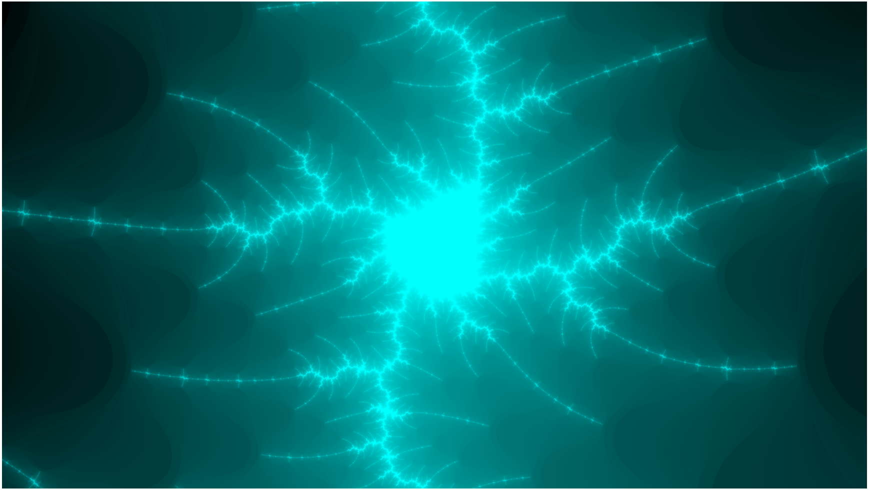

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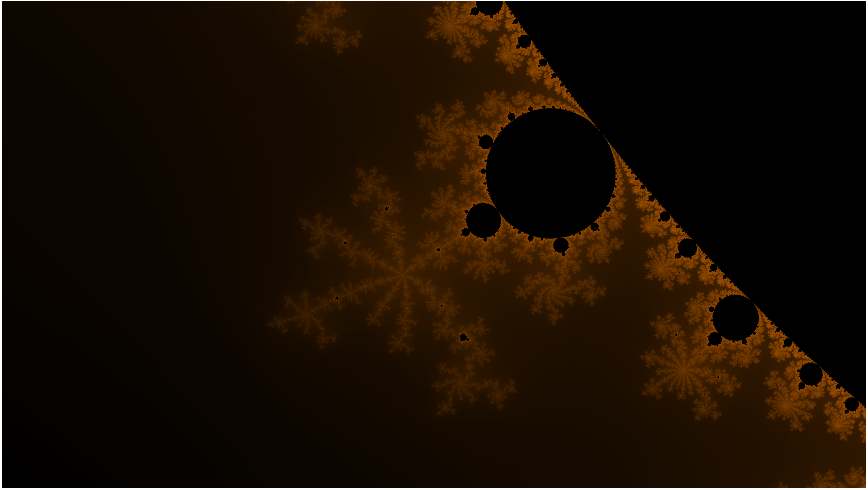

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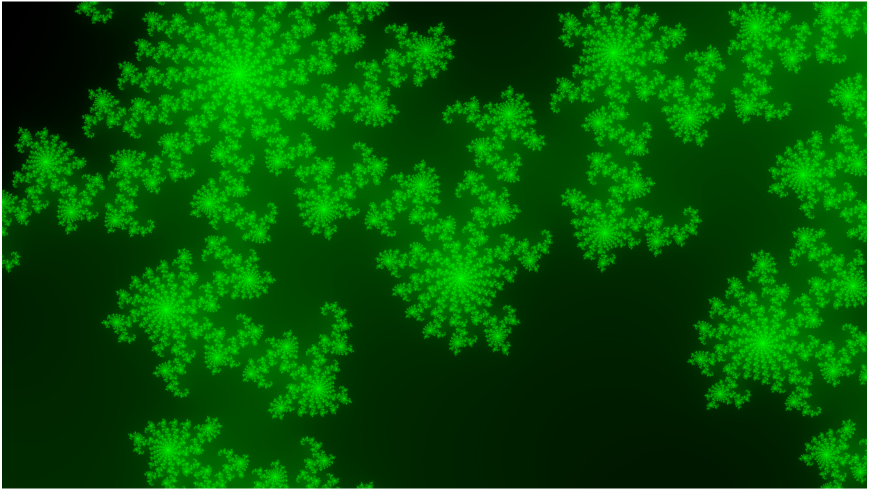

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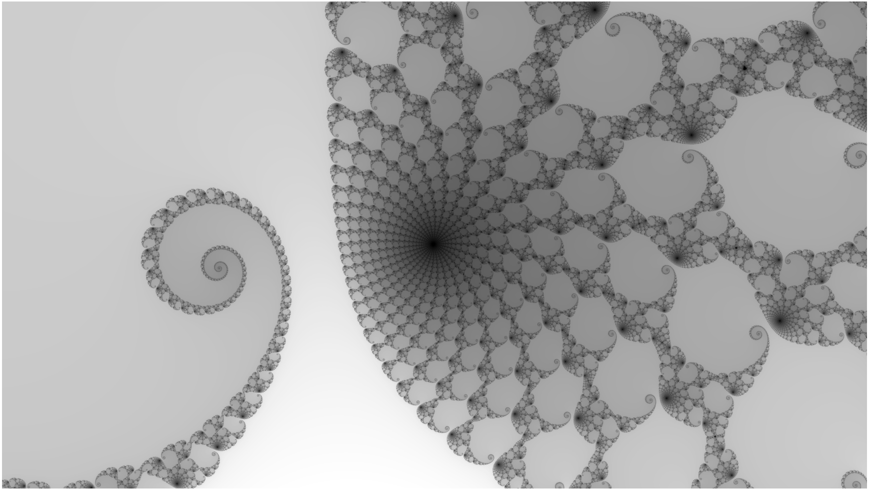

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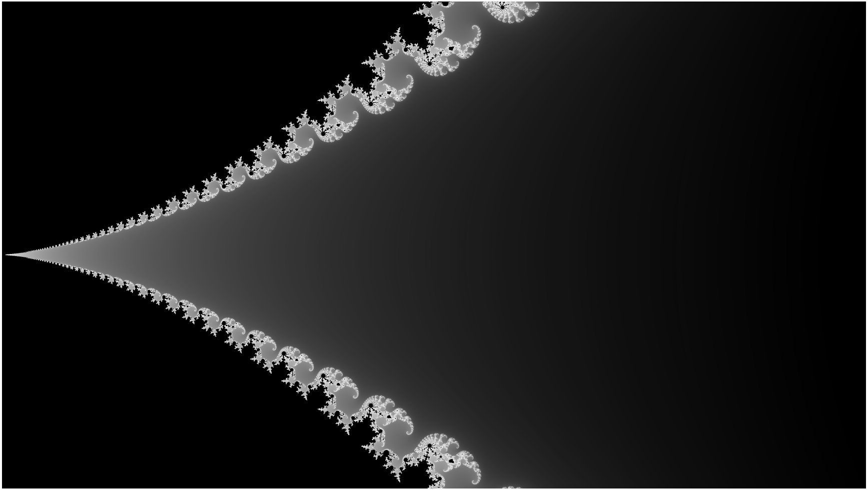

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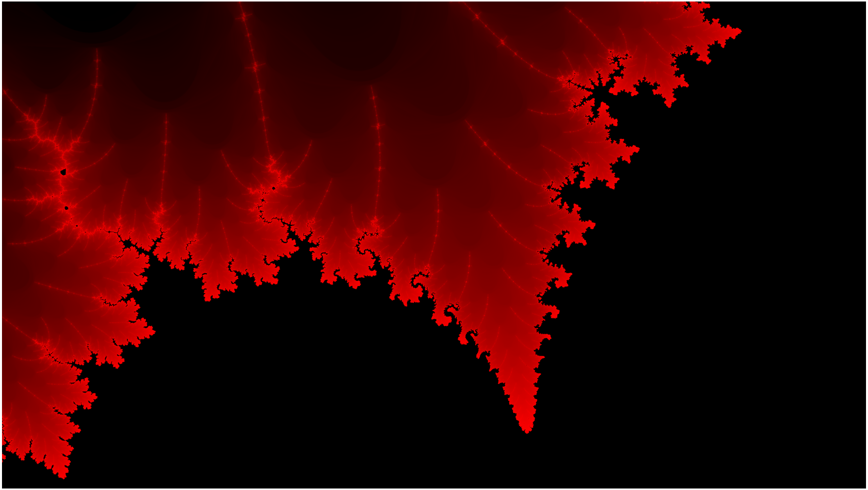

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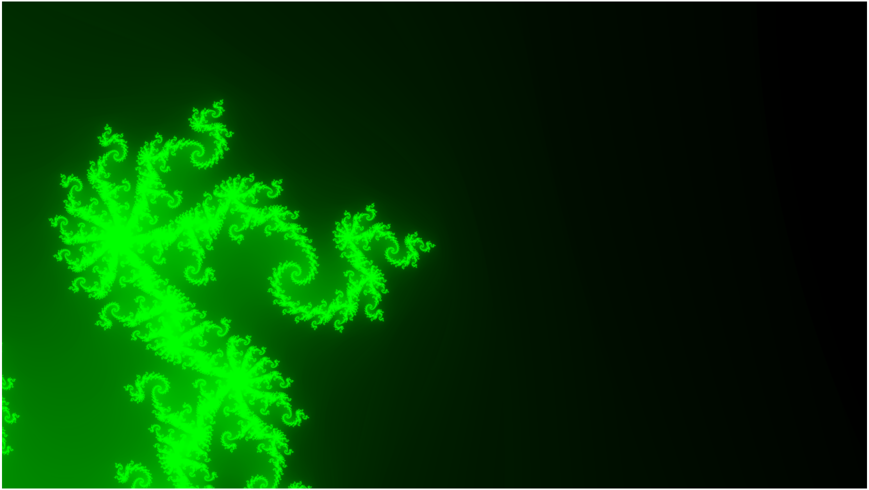

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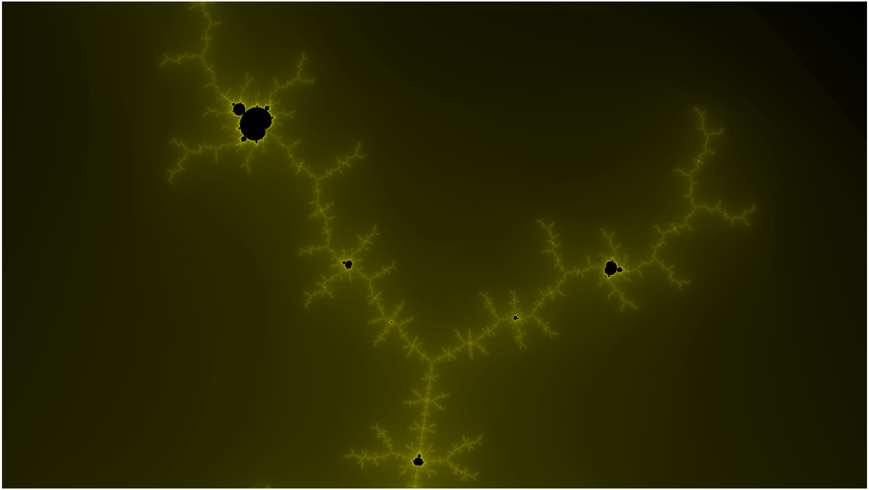

The Mandelbrot Set

The Mandelbrot is all about investigating the following (family of) functions under repeated iteration:

Quadratic Map

$f_c(z) = z^2 + c,\ c \in \mathbb{C}$

The Mandelbrot Set and Example Orbits

Specifically, we investigate what happens when we plug in 0 to f_c, and iterate. For some map f_c, we call the sequence we get the *orbit* of 0.

$$(0, f_c(0), f_c(f_c(0)), \ldots)$$

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Or, as a recursive sequence,

$$\begin{aligned} z_{n+1} &= f_c(z_n) \\ z_0 &= 0 \end{aligned}$$

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Or, as a recursive sequence,

$$z_{n+1} = z_n^2 + c$$
$$z_0 = 0$$

Note that this sequence depends **only** on c.

The Mandelbrot Set and Example Orbits

What does this sequence look like? Let's let $c = 0$.

$$z_0 = 0$$

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$$z_1 = 0$$
$$z_2 = z_1^2 + c$$

What does this sequence look like? Let's let $c = 0$.

$$
\begin{aligned}
z_0 &= 0 \\
z_1 &= 0 \\
z_2 &= 0^2 + 0
\end{aligned}
$$

What does this sequence look like? Let's let $c = 0$.

$$\begin{aligned} z_0 &= 0 \\ z_1 &= 0 \\ z_2 &= 0 \\ &\vdots \end{aligned}$$

The Mandelbrot Set and Example Orbits

What does this sequence look like? Let's let $c = 0$.

$$\begin{aligned} z_0 &= 0 \\ z_1 &= 0 \\ z_2 &= 0 \\ &\vdots \end{aligned}$$

Clearly $z_n = 0$ for all n. So z_n stays bounded for $c = 0$.

The Mandelbrot Set and Example Orbits

Let's let $c = 1$.

$$0$$

The Mandelbrot Set and Example Orbits

Let's let $c = 1$.

$$0 \xmapsto{0^2+1} 1$$

Let's let $c = 1$.

$$0 \xmapsto{0^2+1} 1 \xmapsto{1^2+1} 2$$

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$$0 \xmapsto{0^2+1} 1 \xmapsto{1^2+1} 2 \xmapsto{2^2+1} 5$$

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$$0 \xmapsto{0^2+1} 1 \xmapsto{1^2+1} 2 \xmapsto{2^2+1} 5 \xmapsto{5^2+1} 26$$

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And so we can see that z_n gets arbitrarily large for $c = 1$.

The Mandelbrot Set and Example Orbits

The Mandelbrot set is defined to be all points $c \in \mathbb{C}$ such that the sequence $\{f_c^n(0)\}$ stays bounded.

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The Mandelbrot Set

$M = \{c \in \mathbb{C} : \{f_c^n(0)\} \not\to \infty\}$

(Technically, $z_n \not\to \infty$ is not strictly the same as z_n being unbounded, but since f_c is a polynomial they are equivalent.)

To generate the image of the Mandelbrot set, we colour a point in black if it is in the set. If a point is not in the set, we give it a colour based on how long it takes to diverge.

Example Orbits

Back to orbits, we so far have seen that when $c = 1$, z_n diverges to infinity, and when $c = 0$, z_n stays fixed at 0.

Example Orbits

Back to orbits, we so far have seen that when $c = 1$, z_n diverges to infinity, and when $c = 0$, z_n stays fixed at 0.

There are *many* other ways a point can stay stable. We will go through a few more examples.

Example Orbits

Let's let $c = -1$.

$$z_0 = 0$$

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$$0 \xmapsto{0^2-1} -1$$

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$$0 \xmapsto{0^2-1} -1 \xmapsto{(-1)^2-1} 0 \xmapsto{0^2-1} -1 \cdots$$

Another way to write this:

$$\begin{tikzcd} 0 \arrow[r, bend left, mapsto, "0^2-1"] & -1 \arrow[l, bend left, mapsto, "(-1)^2-1"] \end{tikzcd}$$

Example Orbits

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Another way to write this:

$$\begin{tikzcd} 0 \arrow[rr, mapsto, bend left, "0^2-1"] & & -1 \arrow[ll, mapsto, bend left, "(-1)^2-1"] \end{tikzcd}$$

So z_n cycles between 0 and 1. Namely, it stays bounded and hence is in the Mandelbrot set.

(This is NOT a commutative diagram! Do NOT get your hopes up category theory fans!)

Example Orbits

Let's let $c = i$.

$$0$$

Example Orbits

Let's let $c = i$.

$$0 \longmapsto i$$

Example Orbits

Let's let $c = i$.

$$0 \longmapsto i \longmapsto -1 + i$$

Example Orbits

Let's let $c = i$.

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Example Orbits

Let's let $c = i$.

$$0 \longmapsto i \longmapsto -1+i \longmapsto -i \longmapsto -1+i \longmapsto -i \cdots$$

So, eventually, z_n cycles between $-i$ and $-1+i$. So, we can redraw the diagram.

$$\begin{tikzcd} 0 \arrow[r, mapsto] & i \arrow[r, mapsto] & -1+i \arrow[r, mapsto, bend left] & -i \arrow[l, mapsto, bend left] \end{tikzcd}$$

Here, z_n is not immediately in a cycle, but it falls into one. Since it stays stable, i is also in the Mandelbrot set.

Example Orbits

So far, we the sequences we have investigated have been cycles, or eventually fallen into a cycle. However, most values do not fall into an exact cycle.

Let's look at $c = 0.25$

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$$0.25 \longmapsto 0.3125 \longmapsto 0.3477 \longmapsto 0.3708 \dashrightarrow 0.4896 \longmapsto 0.4897 \dashrightarrow 0.4993 \cdots$$

Here we can see that z_n seems to tend towards 0.5.

Let's start exploring these orbits more visually. Using Desmos!

Investigating Orbits with Julia Sets

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We are going to fix $c \in \mathbb{C}$, and instead of investigating the orbit of 0, investigate the orbit of an arbitrary point z.

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So why do the orbits look like that? Why do the orbit seem to converge if c is in the main cardioid? And why do we get these other patterns in the bulbs?

We are going to fix $c \in \mathbb{C}$, and instead of investigating the orbit of 0, investigate the orbit of an arbitrary point z.

If we know what f_c does to any z, then we can see what it does to z_1, then see what it does to z_2, and etc.

Investigating Orbits with Julia Sets

Let's go back to Desmos, and this time fix $c \in \mathbb{C}$, and look at the orbits of arbitrary z (instead of $z = 0$).

Investigating Orbits with Julia Sets

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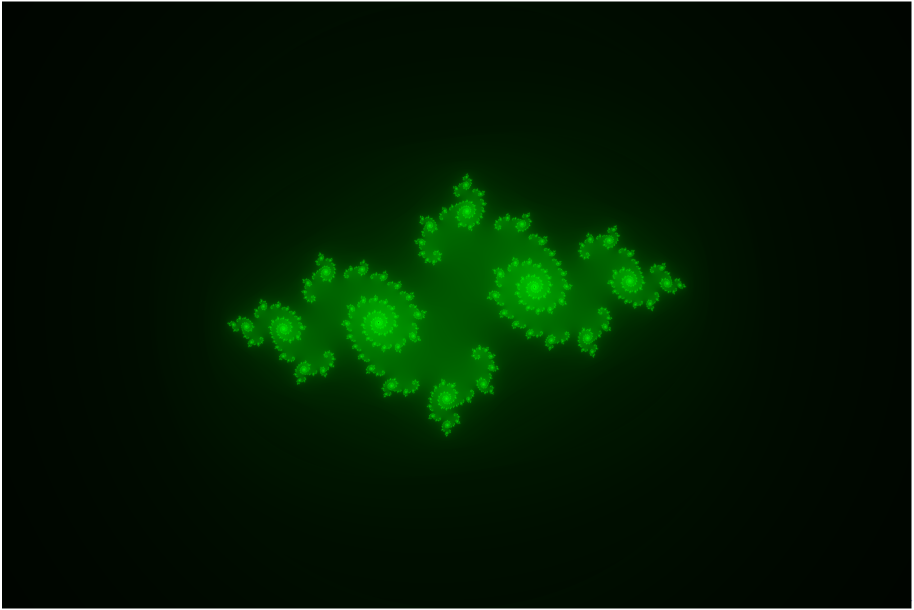

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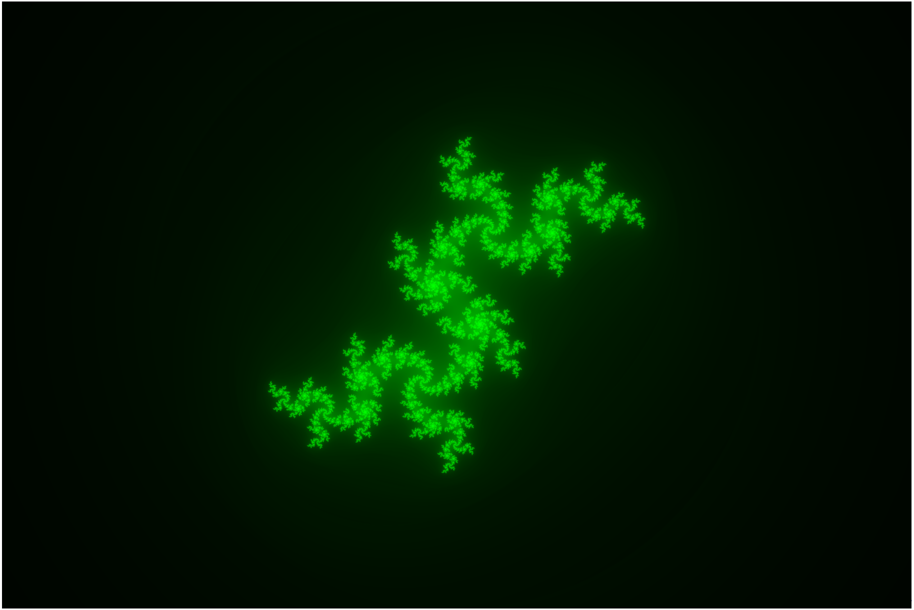

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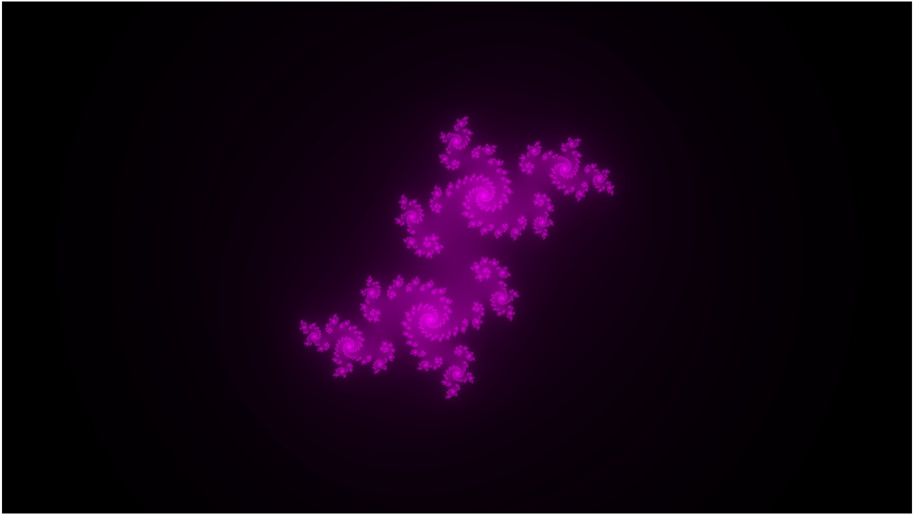

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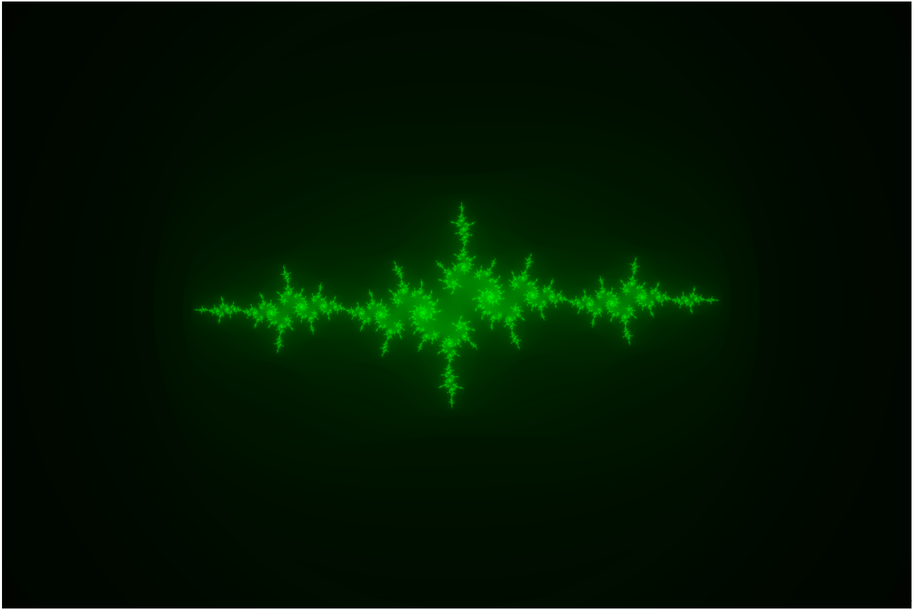

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Transitioning from the Mandelbrot set to the Julia sets was quite subtle since we have the exact same equation for both.

$z^2 + c$	c	z
Mandelbrot		
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Anyway, back to the orbits of points in the Julia sets.

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2. This evolution depends on the Julia set (choice of c), and converges either to a point, or jumps between multiple points.

Let's fix $c \in \mathbb{C}$ and assume that we converge to something. Let's see if we can understand what the orbits converge to, and why they converge to it.

Attractive Fixed Points

Let's call the limit z^*.

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$$\begin{aligned} f_c(z^*) &= z^* \\ (z^*)^2 + c &= z^* \\ (z^*)^2 - z^* + c &= 0 \\ \implies z^* &= \frac{1 \pm \sqrt{1-4c}}{2} \end{aligned}$$

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The negative root turns out to be the one we are looking for. So, in general for $f_c(z) = z^2 + c$, we have the fixed point of interest is given by:

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Let's see what this looks like as we change c in the Mandelbrot set. (In Desmos!)

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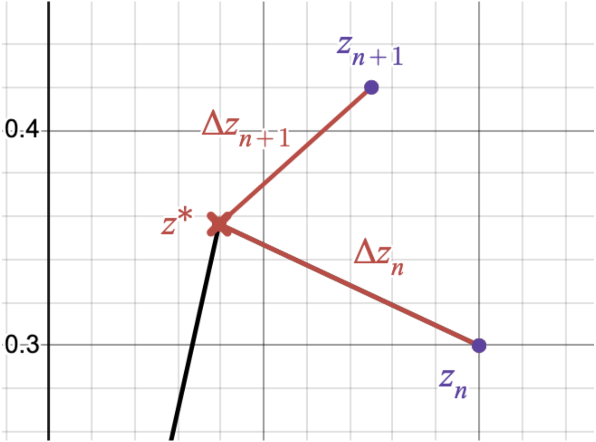
In particular, we shift the origin to z^*. We let Δz_n be such that $z_n = z^* + \Delta z_n$. We try to see how Δz_n is affected by an iteration of f_c.



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$$z_{n+1} = f_c(z^* + \Delta z_n) \approx f_c(z^*) + f_c'(z^*)\Delta z_n$$

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$$\implies \Delta z_{n+1} \approx f_c'(z^*)\Delta z_n$$

So nearby to z^*, the orbit is rotated and scaled by a constant amount (relative to z^*) each iteration.

Importantly, we have that if $|f_c'(z^*)| < 1$, then $\Delta z_n \to 0$, and hence $z_n \to z^*$.

Attractive Fixed Points

This sort of fixed point is *critical* to understanding the orbits of points in the Julia set.

Definition: Attractive fixed point

A point $z_0 \in \mathbb{C}$ is a **fixed point** if

$$f_c(z_0) = z_0$$

This fixed point is called
Attracting if $|f_c'(z_0)| < 1$,
Repelling if $|f_c'(z_0)| > 1$, and
Neutral if $|f_c'(z_0)| = 1$.

Building the Main Cardioid

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We can show this!

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We want to find the set of values c such that f_c has an attracting fixed point. We know that this fixed point must be at $z_0 = \frac{1-\sqrt{1-4c}}{2}$ (The other fixed point is always repelling). At this point,

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$$\begin{aligned} |f_c'(z_0)| &= 1 \\ \implies 1 - \sqrt{1-4c} &= e^{i2\pi\theta} \\ \implies c &= \frac{e^{i2\pi\theta}}{2}\left(1 - \frac{e^{i2\pi\theta}}{2}\right) \end{aligned}$$

Investigating The Bulbs

Let's now investigate the large circular disk to the left of the main cardioid. Let's go into Desmos and highlight every *second* point of the orbits.

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This means that $f_c^2(z_0) = z_0$. We call z_0 a periodic point (with period 2).

Cycles

In general, we have the following:

Definition: Periodic point of period p

A point $z_0 \in \mathbb{C}$ is said to be a **periodic point of period p of f_c** if

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Definition: p-cycle

If $z_0 \in \mathbb{C}$ is a periodic point of period p, the orbit of z_0 has p elements and is is given by

$$\{f_c^n(z_0)\}_{n=0}^{\infty} = \{z_0, f_c(z_0), f_c^2(z_0), \ldots, f_c^{p-1}(z_0)\}$$

This orbit is called a ***p*-cycle**. The orbit can also be written $\{z_0, z_1, z_2, \ldots, z_{p-1}\}$.

Attractive Cycles

It turns out for a p-cycle, $f_c^{p\prime}(z_0) = f_c^{p\prime}(z_1) = \cdots = f_c^{p\prime}(z_{p-1})$. This means that nearby points will act the same for all points in a cycle.

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Definition: Attracting cycle

Let $\{z_0, z_1, \ldots, z_{p-1}\}$ be a p-cycle. If $|f_c^{p\prime}(z_0)| < 1$, then $\{z_0, z_1, \ldots, z_{p-1}\}$ is an **attracting cycle**.

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Attractive Cycles

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Theorem(Fatou):

Let f be a polynomial. Every attracting cycle of f attracts at least one critical point.

In our case with $f_c = z^2 + c$, we only have one critical point ($z = 0$).

Hence, f_c can have at most one attracting cycle, and if so, the orbit of 0 falls into it.

The Period Bulbs

We found main cardioid as such that for all c in the cardioid, f_c has an attractive fixed point. (i.e. the orbit of 0 under f_c approached a point).

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Similar, the period bulbs are defined such that for all c in a bulb f_c has an attracting q-cycle. Then, the bulb is called a period q bulb.

Building the Period 2 Bulb

We can now play the same game that we did for the main cardioid, but instead applied to period 2 points of f_c.

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We want to find the points $c \in \mathbb{C}$ such that f_c has an attracting 2-cycle. This means there exists z_0 with $f_c^2(z_0) = z_0$, and $|{f_c^2}'(z_0)| < 1$.

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$$0 = z^4 + 2cz^2 - z + c^2 + c$$

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I leave the algebra as an exercise to the audience, but the resulting equation we get is $|z+1| < 1/4$. That is, the circle with radius $1/4$ centred at -1.

Locating the q Bulbs

Can we do the same thing with the period 3 bulbs?

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While we cannot a nice formula for the higher period bulbs, we *can* find where they connect to the Mandelbrot set.

Locating the q Bulbs

Recall that we had the equation for the boundary of the Mandelbrot set,

$$c(\theta) := \frac{e^{2\pi i\theta}}{2}\left(1 - \frac{e^{2\pi i\theta}}{2}\right)$$

Where at this point $c(\theta)$, we have a neutral fixed point with derivative $e^{2\pi i\theta}$.

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This means that at $\theta = p/q$, the linearised dynamics locally looks like a rotation of $2\pi p/q$ (p/q of a full turn).

In the main cardioid, the fixed point is attractive and the cycle is repelling. This bifurcates into a repelling fixed point and attracting q-cycle at $c(p/q)$.

The p/q Bulb

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This means that we have a bulb for every rational number.

There are plenty of super interesting properties we can read off from looking at the bulbs (and their 'spokes').

In fact, You can identify both p and q from the spokes. We have q spokes, and the pth spoke is the smallest.

Additional Curiosities

Theorem(Montel): On any neighbourhood of a point on a Julia set, the orbits of every point in the neighbourhood will fill ALL of $\mathbb{C}$ (except a single point maybe?.)

$$\bigcup_{z \in U} \{f_c^n(z)\}_{n=0}^{\infty} = \mathbb{C} \setminus \{a\}$$

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The Hausdorff dimension of the boundary of the Mandelbrot is 2 (a whole integer above its topological dimension!). It is conjectured to not have an area however.

The minibrots are *dense* on the boundary of the Mandelbrot set. Other names include: Bug, Island, Mandelbrotie, Babybrot, etc

References

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Goodbye

Thanks for watching!