Vector Calculus Without Vectors

Max Orchard

August 29, 2025

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(what MATH2901 could be, maybe)

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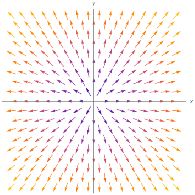

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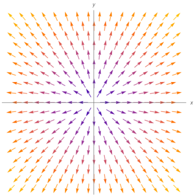

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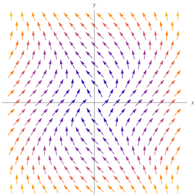

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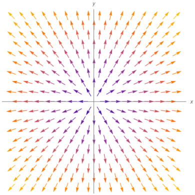

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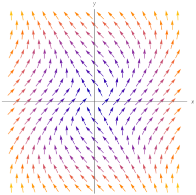

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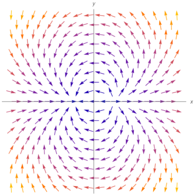

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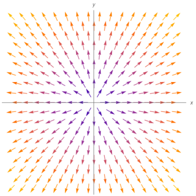

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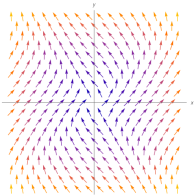

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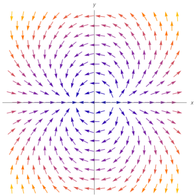

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We will denote the vector space of all vector fields on $\mathbb{R}^n$ as $\mathfrak{X}(\mathbb{R}^n)$.

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Why are there multiple very different ways to differentiate a vector field?

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A *1-form* is a covector field. That is, it is a (smooth) map $\omega : \mathbb{R}^n \to (\mathbb{R}^n \to \mathbb{R})$ that sends $p \in \mathbb{R}^n$ to a covector $\omega_p : \mathbb{R}^n \to \mathbb{R}$.

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We will denote the set of all 1-forms on $\mathbb{R}^n$ as $\Omega^1(\mathbb{R}^n)$.

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Similarly, we denote by $\mathrm{d}y$ the 1-form that takes a vector to its y component:

$$\mathrm{d}y_{(x,y)}(u, v) = v.$$

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$\omega_{(x,y)}(u,v) = y^2u + 2x^2v$ is a 1-form (that is linear in u and v, but not in x and y).

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In general, we cannot multiply 1-forms together and get another 1-form.

Structure of $\mathfrak{X}(\mathbb{R}^n)$ and $\Omega^1(\mathbb{R}^n)$

The spaces $\mathfrak{X}(\mathbb{R}^n)$ and $\Omega^1(\mathbb{R}^n)$ are both vector spaces (with pointwise addition and scalar multiplication):

$$(F + G)(\mathbf{x}) = F(\mathbf{x}) + G(\mathbf{x}), \quad (c \cdot F)(\mathbf{x}) = c \cdot F(\mathbf{x}),$$

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Both vector spaces are n-dimensional, with bases

$$\{\partial x^1, \ldots, \partial x^n\} \text{ for } \mathfrak{X}(\mathbb{R}^n),$$

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Careful: the coefficients are *functions* $F_i(x_1, \ldots, x_n)$, not just scalars!

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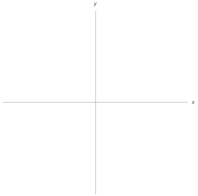

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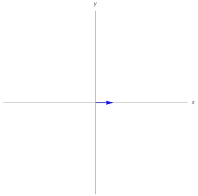

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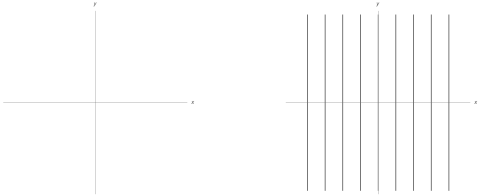

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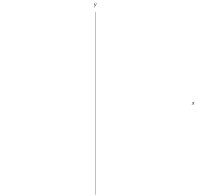

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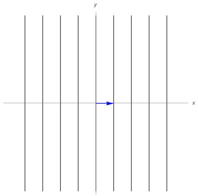

$\mathrm{d}x_{(0,0)}(1,0) = 1$

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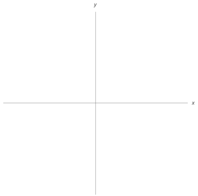

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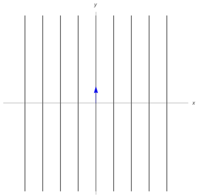

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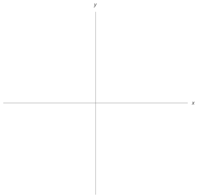

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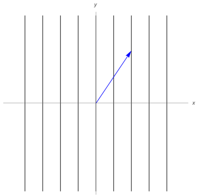

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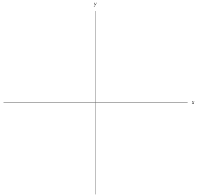

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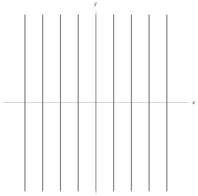

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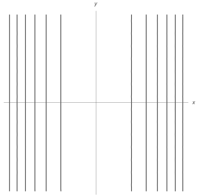

$\omega = x\,\mathrm{d}x$

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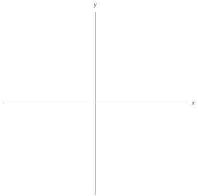

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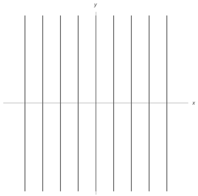

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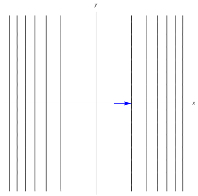

$x\,\mathrm{d}x_{(1,0)}(1,0) = 1$

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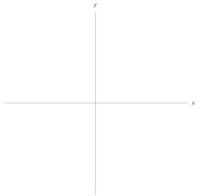

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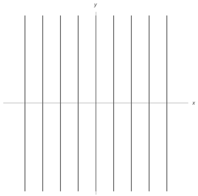

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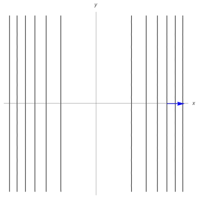

$x\,\mathrm{d}x_{(3,0)}(1,0) = 3$

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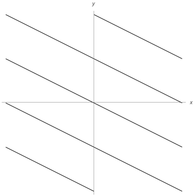

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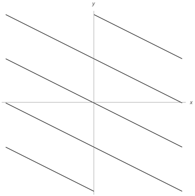

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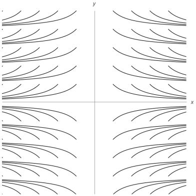

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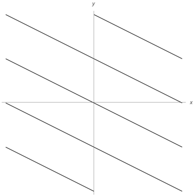

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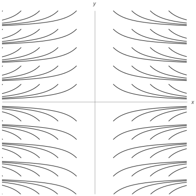

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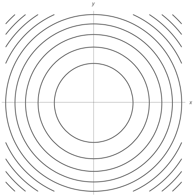

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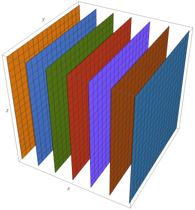

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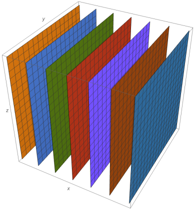

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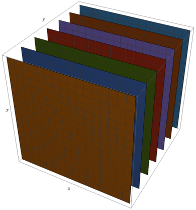

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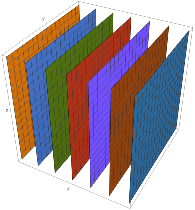

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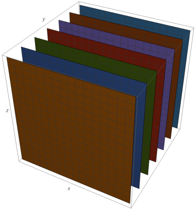

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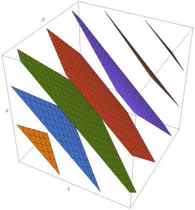

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Vector Field $\leftrightarrow$ 1-Form Correspondence

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The musical isomorphisms give a one-to-one correspondence

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For example, $F(x, y) = (y, x)$ would be identified with $y\,\mathrm{d}x + x\,\mathrm{d}y$.

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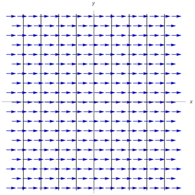

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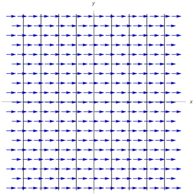

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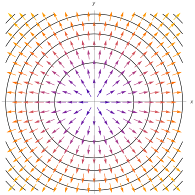

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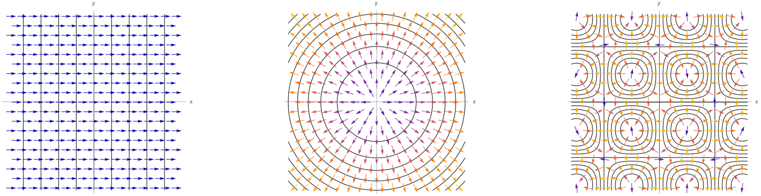

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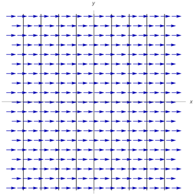

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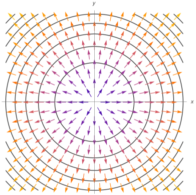

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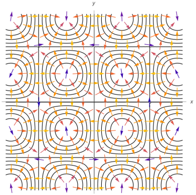

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The inverse operator $\sharp$ can be visualised in a similar way.

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The *gradient* of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the vector field

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This is a vector field, so can be turned into a 1-form through $\flat$. What does this look like?

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Remark: MATH1052/1072 says we have to instead dot with the unit vector $\hat{\mathbf{x}} = \mathbf{x}/\left\|\mathbf{x}\right\|$. We don't do this, otherwise $\mathrm{d}f$ wouldn't be linear.

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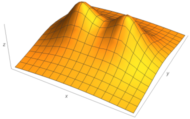

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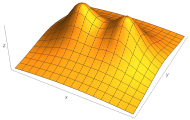

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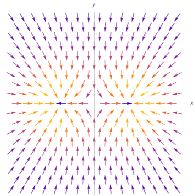

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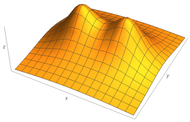

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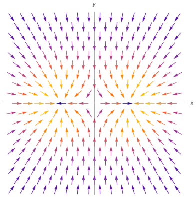

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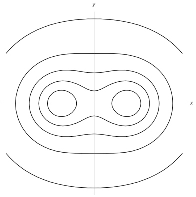

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Enter: the *2-form*.

2-Forms

Definition

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Alternating means $\omega_p(u, v) = -\omega_p(v, u)$.

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A *2-form* assigns to each point an alternating bilinear map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. The space of all 2-forms is denoted $\Omega^2(\mathbb{R}^n)$.

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- If ω_p is bilinear, then ω_p alternating $\iff \omega_p(v, v) = 0$.

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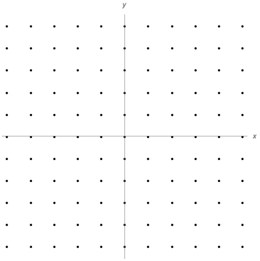

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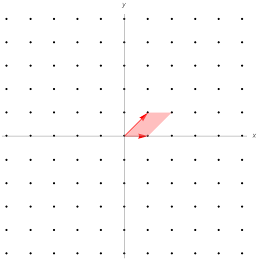

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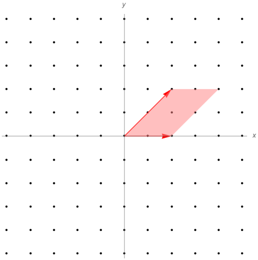

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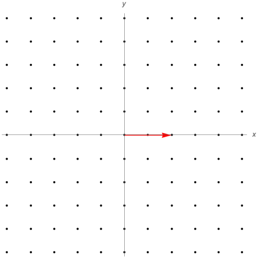

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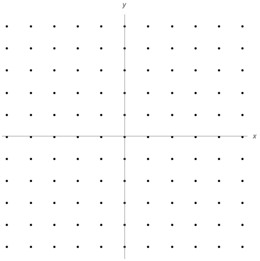

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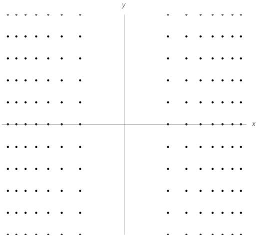

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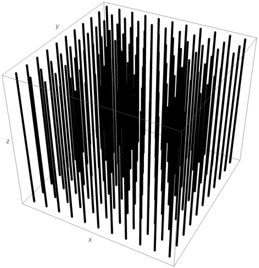

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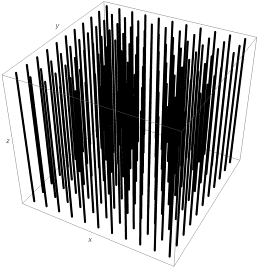

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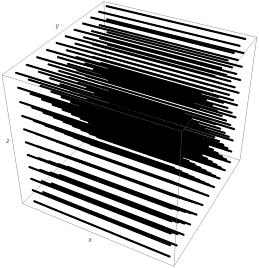

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The Wedge Product

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The *wedge product* is the operator $\wedge : \Omega^1(\mathbb{R}^n) \times \Omega^1(\mathbb{R}^n) \to \Omega^2(\mathbb{R}^n)$ defined by

$$(\alpha \wedge \beta)(\mathbf{x}_1, \mathbf{x}_2) = \alpha(\mathbf{x}_1) \cdot \beta(\mathbf{x}_2) - \alpha(\mathbf{x}_2) \cdot \beta(\mathbf{x}_1).$$

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Note that the definition immediately implies $\alpha \wedge \alpha = 0$.

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It turns out that $\{\mathrm{d}x_i \wedge \mathrm{d}x_j\}_{i<j}$ is a basis for $\Omega^2(\mathbb{R}^n)$. That is, all 2-forms can be generated with just the wedge product alone.

Visualising the Wedge Product

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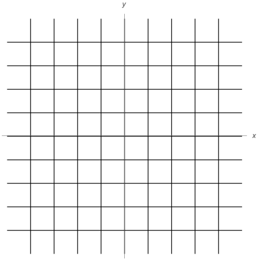

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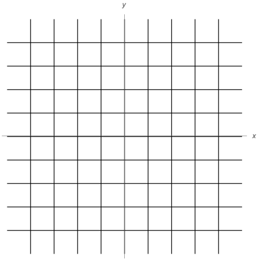

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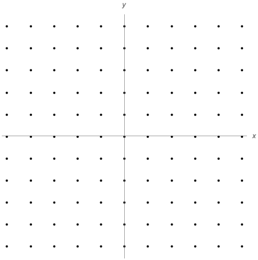

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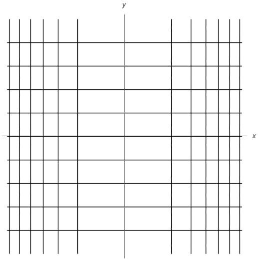

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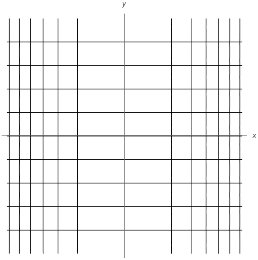

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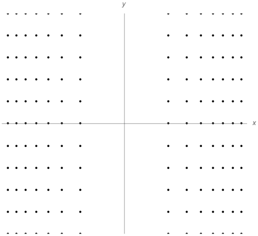

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The Exterior Derivative

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Definition

The *exterior derivative* is the map $\mathrm{d} : \Omega^1(\mathbb{R}^n) \to \Omega^2(\mathbb{R}^n)$ given on multiples of basis forms by

$$\mathrm{d}(f\,\mathrm{d}x_i) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\,\mathrm{d}x_j \wedge \mathrm{d}x_i$$

and extended additively.

Visualising the Exterior Derivative

Intuitively, if $\omega \in \Omega^1(\mathbb{R}^n)$, then $\mathrm{d}\omega(\mathbf{x}, \mathbf{y})$ measures the *difference* in the change in $\omega(\mathbf{y})$ as you move along $\mathbf{x}$ and the change in $\omega(\mathbf{x})$ as you move along $\mathbf{y}$:

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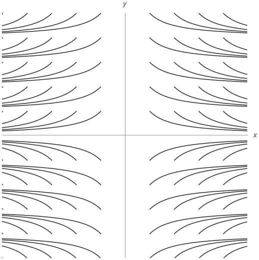

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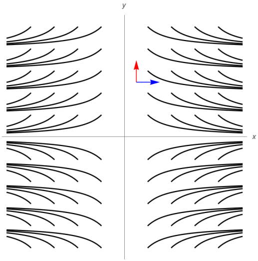

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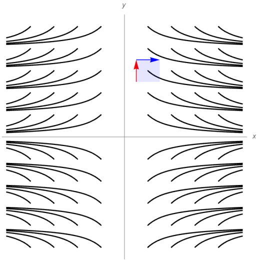

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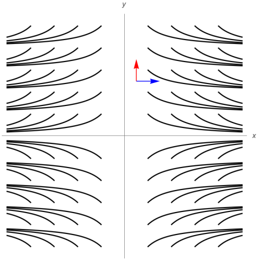

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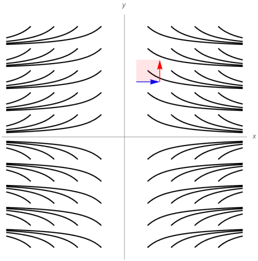

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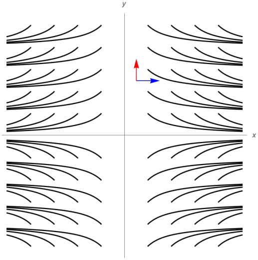

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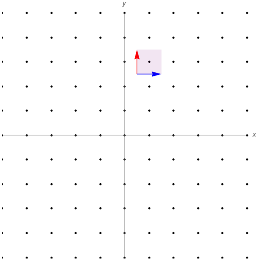

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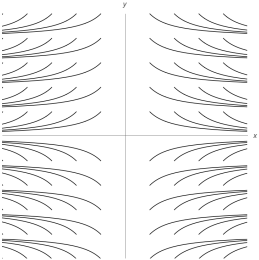

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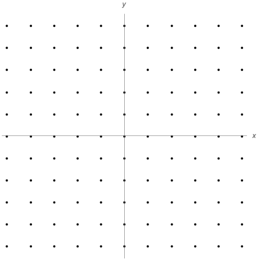

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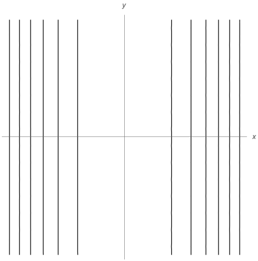

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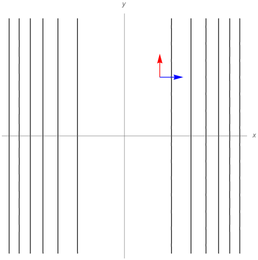

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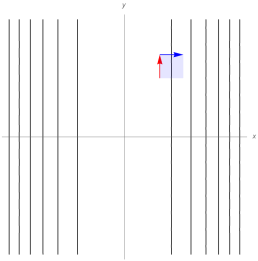

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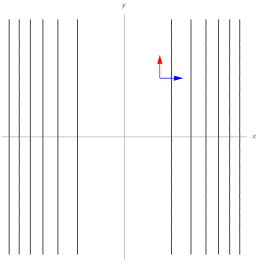

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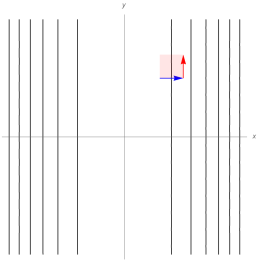

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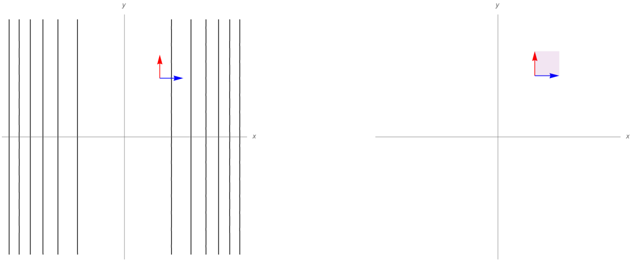
Intuitively, if $\omega \in \Omega^1(\mathbb{R}^n)$, then $\mathrm{d}\omega(\mathbf{x}, \mathbf{y})$ measures the *difference* in the change in $\omega(\mathbf{y})$ as you move along $\mathbf{x}$ and the change in $\omega(\mathbf{x})$ as you move along $\mathbf{y}$:

$$\mathrm{d}\omega(\mathbf{x}, \mathbf{y}) = \mathbf{x}(\omega(\mathbf{y})) - \mathbf{y}(\omega(\mathbf{x})) \quad \text{(for constant } \mathbf{x}, \mathbf{y}\text{)}.$$

The exterior derivative can therefore be seen as the *boundary* of the lines drawn by a form.



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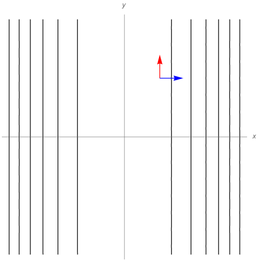

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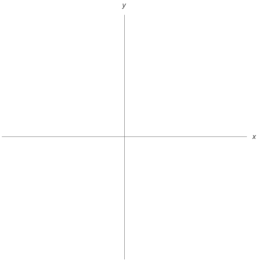

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k-forms

The above constructions can be generalised to *k-forms* (k-linear alternating maps $\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k \text{ times}} \to \mathbb{R}$ assigned to each point).

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The wedge product and exterior derivative extend naturally, though d satisfies a "graded" product rule

$$\mathrm{d}(\omega \wedge \alpha) = (\mathrm{d}\omega) \wedge \alpha + (-1)^{k\ell} \omega \wedge (\mathrm{d}\alpha),$$

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Important fact: by the symmetry of mixed partial derivatives, $\mathrm{d}(\mathrm{d}\omega) = 0$ for all ω. This corresponds to the fact that $\partial(\partial X) = \emptyset$.

Structure of $\Omega^k(\mathbb{R}^n)$

As we've come to expect, a basis for $\Omega^k(\mathbb{R}^n)$ is given by

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Remark: A scalar field $f : \mathbb{R}^n \to \mathbb{R}$ can be thought of as taking in zero vectors. Because of this, we say scalar fields are *0-forms*, and denote $C^\infty(\mathbb{R}^n, \mathbb{R}) = \Omega^0(\mathbb{R}^n)$.

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In fact, $\star(\star\omega) = (-1)^{k(n-k)}\omega$.

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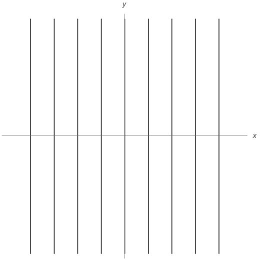

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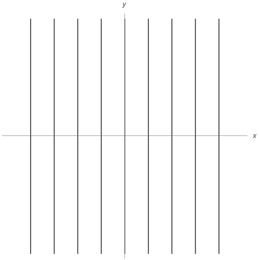

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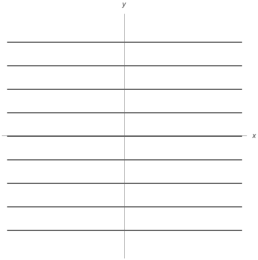

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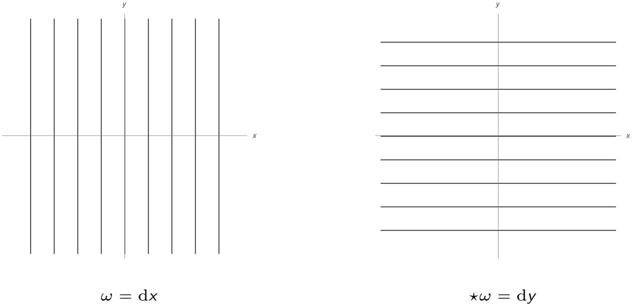

$\omega = \mathrm{d}x$ $\quad\star\omega = \mathrm{d}y$

The fact that $\star(\star\omega) = (-1)^{k(n-k)}\omega$ corresponds to the fact that $(U^\perp)^\perp = U$ (and $U = -U$).

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Let's do some calculus! (Finally!)

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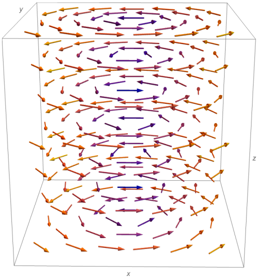

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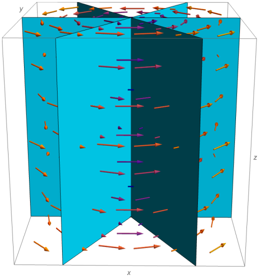

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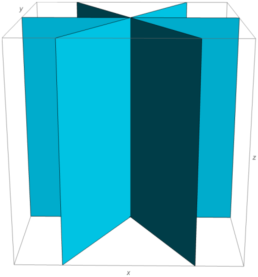

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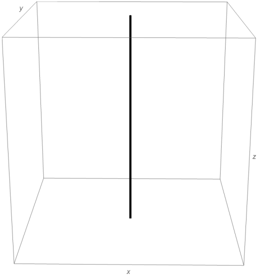

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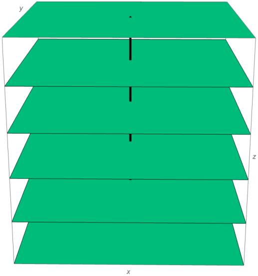

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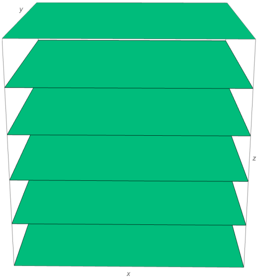

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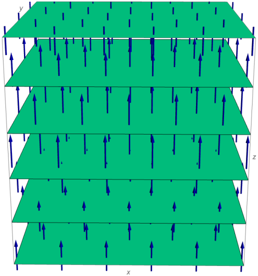
Standard Vector Operations

Let $F \in \mathfrak{X}(\mathbb{R}^3)$, and let's take the exterior derivative of the associated 1-form. That is, let's consider $\mathrm{d}F^\flat$. For this example, I will look at $F(x, y, z) = (-y, x, 0)$.



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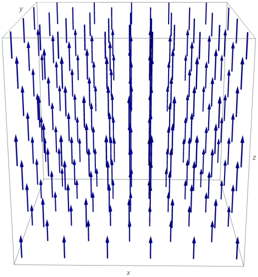

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The $\star$, $\flat$, $\sharp$ kind of obscure what's going on: they are just isomorphisms allowing us to identify one space with another. What we're really doing is *differentiating a 1-form* and interpreting it as a vector field.

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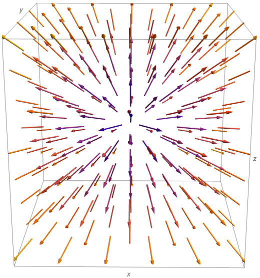

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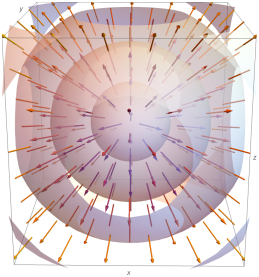

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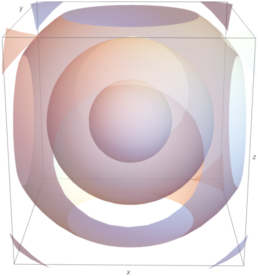

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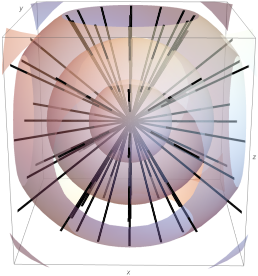

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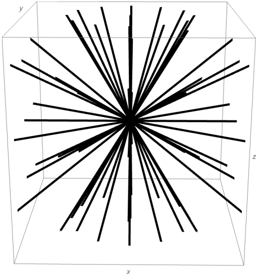

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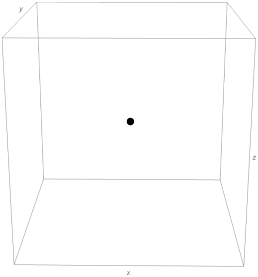

$$\mathrm{d}(\star F^\flat) = 3\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

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Definition

The *divergence* is the operator $\mathrm{div} : \mathfrak{X}(\mathbb{R}^3) \to C^\infty(\mathbb{R}^3, \mathbb{R})$ defined by

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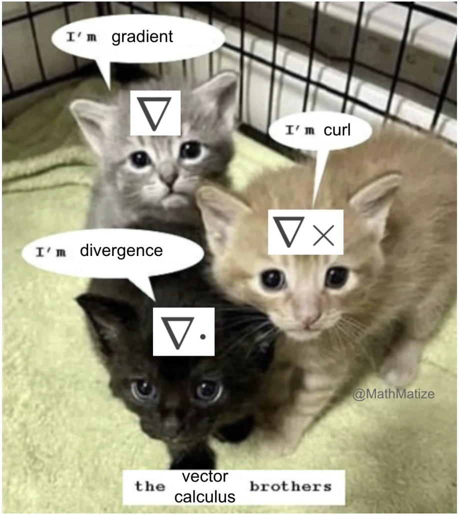
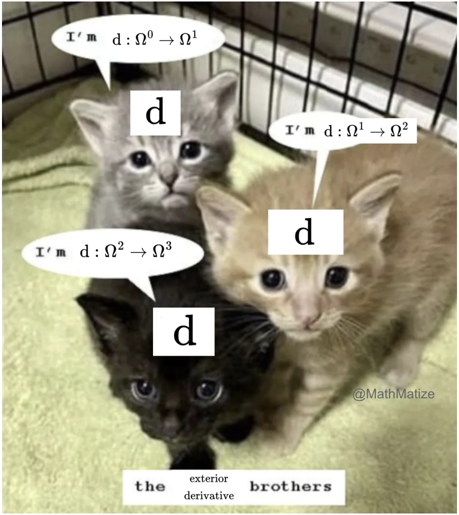
Exercise

We know (MATH2001) that conservative vector fields have zero curl:

$$\nabla \times (\nabla f) = 0.$$

Prove this fact in two lines using the framework of forms. See if you can come up with another similar fact.


I'm gradient
∇
I'm curl
$\nabla\times$
I'm divergence
$\nabla\cdot$
@MathMatize
the vector calculus brothers


I'm $d:\Omega^0 \to \Omega^1$
d
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the exterior derivative brothers

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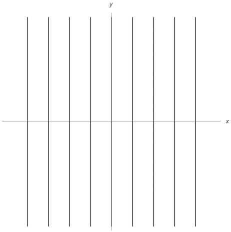

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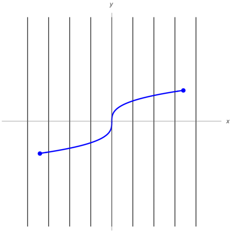

$$\int_\gamma \omega$$

Fundamental Theorem of Line Integrals

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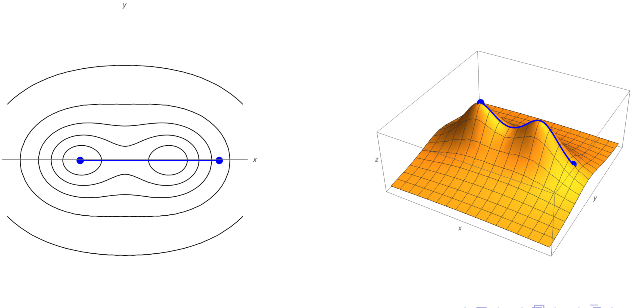

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This is actually saying something very trivial: the number of lines that enter (and don't exit) a region is the number of lines that enter (and don't exit) a region.

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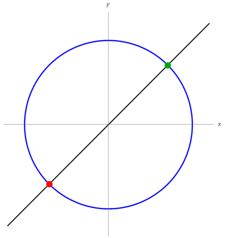

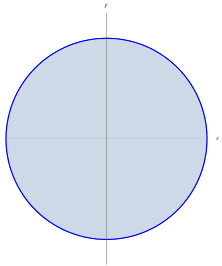

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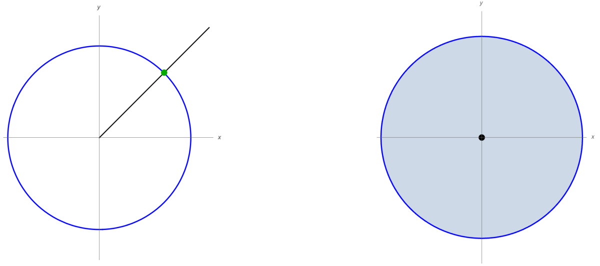

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By varying U and ω, we recover many classical vector calculus theorems.

The Many Faces of Stokes' Theorem

Theorem (Stokes)

We have (with the correct adjectives for $U \subseteq \mathbb{R}^n$)

$$\int_U \mathrm{d}\omega = \int_{\partial U} \omega.$$

By varying U and ω, we recover many classical vector calculus theorems.

U	∂U	ω	$\mathrm{d}\omega$	**Theorem**
$[a, b]$	$\{a, b\}$	f	$f'(x)\,\mathrm{d}x$	FTC
$\gamma([a, b])$	$\{\gamma(a), \gamma(b)\}$	f	$(\nabla f)^\flat$	FTLI
U	∂U	$\omega \in \Omega^1$	$\mathrm{d}\omega$	Green's theorem
S	∂S	$F^\flat$	$\mathrm{curl}(F)^\flat$	Stokes' theorem
V	∂V	$\star F^\flat$	$\star\mathrm{div}(F)$	Divergence theorem

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- the *exterior calculus on manifolds*.
- *closed* and *exact* forms and the *de Rham cohomology*.
- connections and the *Yang–Mills equations*.

References

For some further reading:

- K. Broder. *Lectures on Vector Calculus*, available online at https://www.kylebroder.com/_files/ugd/cb72c5_b2fcc018d79248379cd61faf199ed2bb.pdf, 2022.
- G. Weinreich. *Geometrical Vectors*, The University of Chicago Press, 1998.
- C. Wisner, K. Thorne, J. Wheeler, D. Kaiser. *Gravitation*, Princeton University Press, 2017.