The Power of Choice

Rhea Wolski/#1 Zorn's Lemma Fan AU

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Russel's Paradox (1901):

Let $R = \{x \mid x \notin x\}$. If $R \in R$, then $R \notin R$, but if $R \notin R$, then $R \in R$. Thus $R \in R \iff R \notin R$.

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Zermelo-Fraenkel Set Theory:

"A Set is a Thing that contains Stuff obeying the 8 Zermelo-Fraenkel Axiom Schema"

Zermelo-Fraenkel Axioms

1. Axiom of Extensionality: Two sets are the same if they contain the same elements
2. Axiom of Regularity: A non-empty set contains a member disjoint to it as a set
3. Axiom Schema of Restricted Comprehension: For any set X and any property P, there exists a subset of X: $B = \{x \in X \mid P(x)\}$
4. Axiom of Pairing: For any two sets, there exists a set containing both sets as elements.
5. Axiom of Union: For any set of sets, there exists a set containing every member of the members of the set.
6. Axiom Schema of Replacement: The image of a set under any definable function is a set.
7. Axiom of Infinity: There exists an infinite set.
8. Axiom of Power Set: For any set, there exists a set containing every subset.

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Pretentious Set Theory:

"It is not possible to answer that question in a meaningful way. If I asked you “what is a group element” it doesn't make sense to define it as its stand alone thing. Similarly, it doesn't make sense to ask what is a set as its stand alone thing. Indeed, x can be a set in one model of set theory while not in another model. To see this in the setting of set theory, take any model S of ZFC that is not constructive. Then S has a constructive inner model K. Since K is a proper subclass of S, there is a set x in S that is not in K. Then x is a set in S and not in K. You might then be tempted to say “let's just call sets the collection of all objects in all models of set theory". Two issues with this. Applying same logic to group theory we get that the only group element is e. Secondly, since set theory is foundational, there is no meaningful way to compare two models. You can compare the rigid objects (essentially all sets generated from a finite process from and some countable sets. Formally called 0 definable objects I believe) but that collection is not stable under power-set and thus does not form a valid set theory. It makes more sense to ask what is a set theory (or model of set theory)."

The Axiom of Choice

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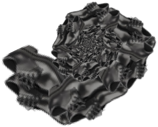

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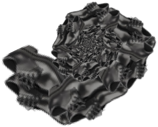

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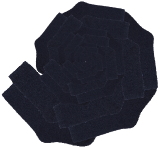

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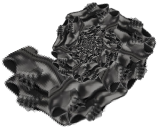

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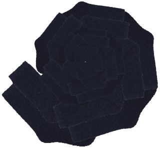

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Axiom of Choice: For any collection X of non-empty sets, there exists a choice function on X.

Zorn's Lemma

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An element m of a Poset such that $\nexists\, s \neq m$ in the Poset satisfying $m \leqslant s$.

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- Every Ring contains a maximal ideal.
- For every infinite set A, there is a bijection to the cartesian product $A \times A$.

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Consider vector spaces such as:

$C(X)$, the space of continuous real-valued functions.

$c_0(\mathbb{R})$, the space of real sequences converging to 0.

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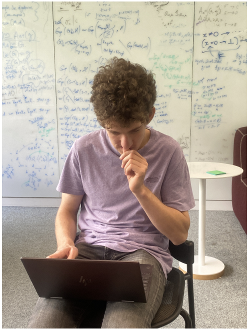

Figure: Total Fucking Sicko

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The Well Ordering Theorem says that every set can be Well Ordered.

This is saying that on every set (in particular, on $\mathbb{R}$), there is a an ordering of all the elements, in which no distinct elements are equal, and any subset has a minimal element. Think about $(0, 1]$ in the reals.

Not an equivalent to AoC, but an implied statement.

Banach Tarski says that we can take a ball, split it into finite pieces, and reassemble it into 2 identical balls to the original.

Figure: I stole this from wikipedia

Idk its kinda weird i guess.

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Group Axioms:

-
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When do we accept axioms?

Figure: Sick ass fucking picture of raven (left) and a pig (right)