A Brief History of Groups

Isaac Beh

Forewarning

- I know very little
- History is messy
- There are many people and events I will skip

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  - The operation is associative
  - There is an identity
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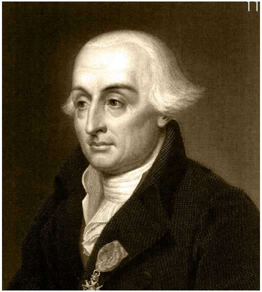
It depends who you ask!

What was a Group to Lagrange?



Joseph-Louis Lagrange
(1736-1813)

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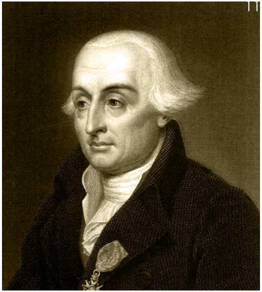

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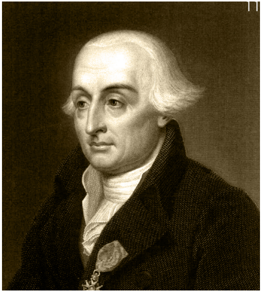

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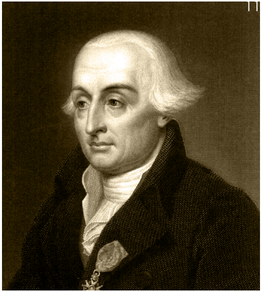
- Let $x_1, \ldots, x_n$ be the roots of some polynomial and $f(x_1, \ldots, x_n)$ is a function.

  How many different values can we get if we permute the order of $x_1, \ldots, x_n$?



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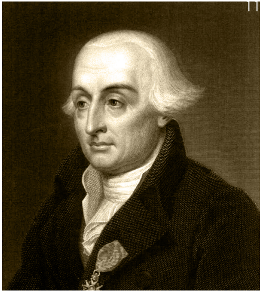
  How many different values can we get if we permute the order of $x_1, \ldots, x_n$?
- For particular functions f, the number of values must divide $n!$



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Évariste Galois
(1811-1832)

What was a Group to Galois?

- A collection of substitutions such that "if in such a group one has the substitutions S and T then one has the substitution ST."
- Defined normal subgroups, simple groups, among many other things

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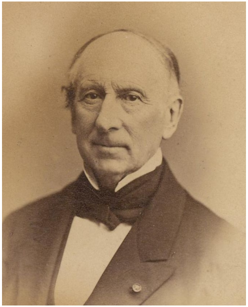
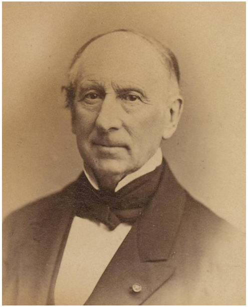


Augustin-Louis Cauchy
(1789-1857)

What was a Group to Cauchy?

- Same permutation-based definition as Galois
- Found all subgroups of S_2, S_3, S_4 and S_5
- Proved Cauchy's theorem:
  If p divides $|G|$ then there is a subgroup of order p



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- Showed that $\mathbb{Z}_n^\times \cong \mathbb{Z}_{\varphi(n)}$

Carl Friedrich Gauss (1777-1855)

What was a Group to Klein

Felix Klein
(1849-1925)

What was a Group to Klein

- "Now let there be a sequence of transformations $A, B, C, \ldots$ If this sequence has the property that the composition of any two ... belongs to the sequence, then this [sequence] will be called a group of transformations"

Felix Klein
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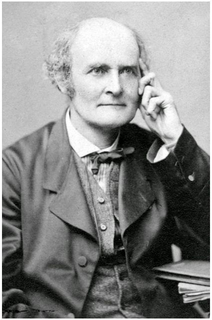

Arthur Cayley
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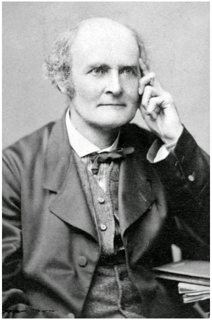
What was a Group to Cayley

- "A set of symbols $1, \alpha, \beta, \ldots$ such that the product of any two of them ... belongs to the set, is said to be a group.



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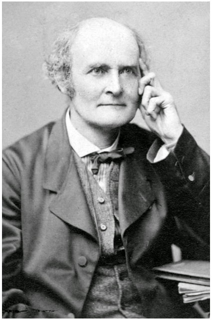

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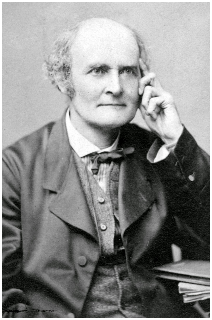

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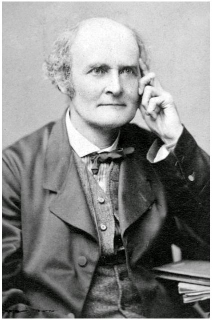

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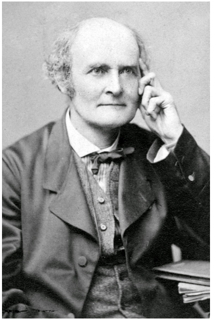
- "... the object of law was to say a thing in the greatest number of words, of mathematics to say it in the fewest."



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What Comes Next

1950 Kronecker proved the classification of finite abelian groups

1872 Sylow proved the Sylow theorems

1894 Cartan proved the classification of semisimple Lie algebra, and the classification of simple Lie groups

1890's Frobenius and Burnside work on the representation theory for groups

1950–83 The classification of finite simple groups is completed

Burnside's Lemma

If G acts on X then

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

~~Burnside's~~ Frobenius' Lemma

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The Lemma That Is Not Burnside's

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Interesting Things To Read

- The Evolution of Group Theory: A Brief Survey by Israel Kleiner
- A Hundred Years of Finite Group Theory by Peter M Neumann in The Mathematical Gazette (1996)
- Galois Theory by Ian Stuart
- Convolutions in French Mathematics (1800–1840) by Ivor Grattan-Guinness
- Mathematicians: The History of Math Discoveries Around the World by Leonard C. Bruno