Colouring Knots with Primes

Ed Hawkins (any/all)

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Disclosures

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4. Saying "Knot" and "Not" is confusing...

Knots

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New (Better) "Definition"

A knot is a line (piece of string) we move around in 3D space, and then glue its ends together. We do not care about the length of said line, only the way in which it becomes "knotted" (I'm not defining what I mean by "knotted").

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This gives us a convenient way to represent knots, without a mess of (possibly unclear) instructions as to how to tie them.

Examples of Knots

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Trefoil.png

Figure: Knot 3_1 ; the Trefoil

Examples of Knots

Figure 8 Knot.png

Figure: Knot 4_1 ; the Figure-Eight knot

Examples of Knots

Unknot.png

Figure: Knot 0_1 ; the Unknot

Knot Examples

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But we need to introduce a few more tools and ideas before we can.

Knot Equivalence

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Definition: Ambient Isotopy

We say two knots are equivalent if we can continuously transform one into the other without cutting the line, or passing though itself; i.e. can we "untie" them until they look the same? If so, we then say they are **Ambient-Isotopic** to one another.

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You might say that saying we can "continuously transform one into the other" is a vague definition. And you'd be right.

Reidemeister Moves

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Reidemeister Moves.png

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Reidemeister moves example

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Figure 8 Knot transformation.jpeg

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We importantly care about the contrapositive of this statement, being that if $\varphi(K_1) \neq \varphi(K_2)$ then K_1 and K_2 are different knots.

Knot Colourability

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Colourability

We say a knot diagram is colourable if we can assign each strand one of three labels (colours) such that the following hold:

1. We use at least two distinct colours, and
2. At any crossing, if two colours appear, then all three colours appear.

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We now prove that this is infact a knot invariant.

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Figure: Coloured Trefoil

So since we've shown that colourability is a knot invariant, we now know the trefoil and the unknot are actually different knots.

Yippee.png

Primes

Recall our definition of colourability:

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Primes

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Colourability

We say a knot diagram is colourable if we can assign each strand one of three labels (colours) such that the following hold:

1. We use at least two distinct colours, and
2. At any crossing, if two colours appear, then all three colours appear.

Observe that if we colour our strands with elements from $\{0, 1, 2\} = \mathbb{Z}_3$, condition (2) is equivalent to:

For $x, y, z \in \mathbb{Z}_3$, at any crossing with over strand-coloured z, and under-strands coloured x and y,

$$x + y \equiv 2z \pmod 3$$

Primes

p-Colourability

We say a knot diagram is p-colourable for some prime p$\geq$ 3 if we can label (colour) it using elements from $\mathbb{Z}_p$ such that:

1. At least two distinct elements are used, and
2. At any crossing with over-strand coloured z and under-strands coloured x and y,

$$x + y \equiv 2z \pmod{p}$$

This is a knot invariant $\forall$ primes p $\geq$ 3 (trust)

p-Colourable Example

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Figure 8 Knot 5 coloured.png

Figure: The Figure-8 Knot is 5-colourable

Matrices

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We can represent this system with an $n \times n$ coefficient matrix, with rows corresponding to crossings, and the columns representing the strands.

We construct such a matrix carefully because of the unique case where the over strand is one of the under strands.

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2. At the ith crossing, in the corresponding row, add 2 to the over strand entry, and subtract 1 from each of the under strand entries.
3. Repeat $\forall\ i \in \{1, 2, 3, ..., n\}$
4. Remove one arbitrary row and column

This actually **isn't** a knot invariant, but the following is...

Knot Determinants

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Given a knot K and a corresponding matrix constructed as above M_K, we define the Knot Determinant of K as

$$\det(K) = |\det(M_K)|$$

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In fact, there's a nice theorem that relates p-colourability and these determinants.

But first, lets do an example.

Knot Determinate Example

Labeled Trefoil.png

Figure: 3_1; the Trefoil

1. We label a knot and its crossings, and create a 3×3 matrix of all 0's:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Knot Determinate Example

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Figure: 3_1; the Trefoil

2. We add 2 to each over-strand entry:

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

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3. And add -1 to each under-strand entry:

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

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Figure: 3_1; the Trefoil

4. We remove an arbitrary row and column, and compute the determinate:

$$\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$$\therefore \det(3_1) = 3$$

P-Colourability and Knot Determinant Theorem

Theorem

A knot K is p-colourable for primes $p \geq 3$ **iff** $p \mid \det(K)$

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This theorem actually isn't too hard to prove, if you agree that the matrix M_K is an accurate representation of the desired system of equations.

P-Colourability and Knot Determinant Theorem

Theorem

A knot K is p-colourable for primes $p \geq 3$ **iff** $p | \det(K)$

This theorem actually isn't too hard to prove, if you agree that the matrix M_K is an accurate representation of the desired system of equations.

Which isn't immediately obvious :(

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Since we are looking for solutions $\mathrm{mod}\ p$, and a matrix A has non-trivial null-space iff $\det(A) = 0$,

$$\det(M_K) \equiv 0 \pmod{p} \iff p | \det(M_K) \iff p | \det(K)$$

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$\square$

(At least) Countably infinite knots

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Proving (at least) countably infinite knots

Consider a knot with $n \geq 3$ crossings s.t. n is odd, and each strand is the over-strand of **exactly** one crossing that is not said strands end.

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The $n \times n$ matrix corresponding to such a knot is given by:

$$\begin{pmatrix} 2 & -1 & 0 & & 0 & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ -1 & 0 & 0 & & 0 & -1 & 2 \end{pmatrix}$$

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We choose to remove the n$^{\text{th}}$ row and column to make our lives easier.

Proving (at least) countably infinite knots

This leaves us with the $n-1 \times n-1$ matrix:

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Which has a determinate of n, which can be proven with generating functions or induction.

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Which has a determinate of n, which can be proven with generating functions or induction.

Now we need only show that such a knot exists for each odd $n \geq 3$, since the determinate of our knots is an invariant.

Intermediate Definition

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Definition: Braid

A **Braid** is a set of n strings which can be interpreted as attached to a horizontal bar at each end. Each string is positioned such that it intersects a horizontal plane exactly once.
By connecting each of the strands on the top bar with corresponding strands on the bottom bar, we obtain a knot or link.
For a knot K obtained this way, we say the braid is the *braid representation* of K.

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For a knot K obtained this way, we say the braid is the *braid representation* of K.

Ok now we can get back to our proof.

Back to Proving (at least) countably infinite knots

We show such a knot exists using the following braid, which is just a fancy way of representing a knot.

Braid.png

Figure: Braid 1

Back to Proving (at least) countably infinite knots

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We attach n of this braid together "on-top" of each other.

Proving (at least) countably infinite knots

n-braid.png

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So now you know:

1. What a knot is, and what it means for two of them to be the same
2. What it means for two knots to be different, and some tools to help determine this
3. That there are (at least) a countably infinite number of knots

References

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