Homotopy Type Theory 2:

Electric Boogaloo

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26 April 2024

Introduction

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I understand the basic concepts of HoTT. I do not actually understand homotopy theory (a different subject).

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But what if G is actually a ring? **Bad question!**
The typing "relation" isn't a proposition. A mathematical object is *always* associated with its type, by its very nature.

Dependent Function Types

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This can be generalized to allow the type of the codomain to *depend* on an element of the domain. For example, the type of "identity matrix" (with entries from $\mathbb{R}$) is

$$I : (n : \mathbb{N}) \to M_{n\times n}(\mathbb{R}) \qquad \Big(I : \prod_{n:\mathbb{N}} M_{n\times n}(\mathbb{R})\Big).$$

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This yields such believable results as $I_3 : M_{3\times 3}(\mathbb{R})$.

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This can be generalized to a dependent pair type. For example, to represent arbitrary tuples of rational numbers:

$$\left(3, \left(\frac{1}{69}, -\frac{1}{420}, 666\right)\right) : (n : \mathbb{N}) \times \mathbb{Q}^n \qquad \left((0, ()) : \sum_{n:\mathbb{N}} \mathbb{Q}^n\right).$$

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Often, U_i is used instead of Type_i (note that i is a *meta-theoretic* natural number).
It is a bad idea to set up a type theory where $\mathrm{Type} : \mathrm{Type}$, due to the type-theoretic version of Russel's paradox (Girard's paradox).

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- Universal quantification is the dependent function type: $(x : A) \to P(x)$.
- Disjunction and existential quantification are best discussed after *propositional truncation* has been introduced.

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The identity type is also called the path type; the homotopical intuition is that the terms of $a =_A b$ are like paths from a to b in the space A.

Using Equality

Functions out of the identity type can be defined by *path induction* (briefly explained later). Thusly, equality can be proved to be symmetric and transitive: for any type A,

$$\begin{aligned} \mathrm{sym} &: (a, b : A) \to a = b \to b = a, \\ \mathrm{trans} &: (a, b, c : A) \to a = b \to b = c \to a = c. \end{aligned}$$

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These proofs of symmetry and transitivity are not just mere properties, but themselves have structure (in this case, that of an ∞-groupoid, a higher-categorical version of a group).
Some other useful operations: for any types A, B,

$$\begin{aligned} \mathrm{ap} &: (f : A \to B) \to (x, y : A) \to x = y \to f(x) = f(y), \\ \mathrm{transport} &: A = B \to A \to B. \end{aligned}$$

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Given a type A, and $x, y : A$, we define the theorem of *indiscernability of identicals*

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One may also define the *identity of indiscernables*:

$$\begin{aligned}\text{indis-iden} &: ((P : A \to \text{Type}) \to P(x) \to P(y)) \to x = y \\ \text{indis-iden}\ h &:= h\ (z \mapsto x = z)\ \text{refl}(x)\end{aligned}$$

Inductive Types

Where do types like $\mathbb{N}$ come from? They are “freely generated” from “constructors”. The “inductive definition” of $\mathbb{N}$ is

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Despite the high quotation mark-density, this can be made precise; the validity of similar inductive definitions can be checked by a mechanical process.

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Defining a function on the natural numbers by recursion, and proving a property of natural numbers by induction, are special cases of this "elimination principle".
Explicitly, this elimination principle can be expressed in type theory itself as:

$$\begin{aligned}\mathbb{N}\text{-elim} : (P : \mathbb{N} \to \mathrm{Type}) \to P(0) \to ((n : \mathbb{N}) \to P(n) \to P(S(n)))\\ \to (n : \mathbb{N}) \to P(n).\end{aligned}$$

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Here is a definition by *pattern matching* (equivalent to eliminator use) of $+$ on $\mathbb{N}$:

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Here is an inductive proof that $+$ is associative:

$$\begin{aligned} \text{+-is-assoc} &: (a, b, c : \mathbb{N}) \to a + (b + c) = (a + b) + c; \\ \text{+-is-assoc}\ a\ b\ 0 &:= \mathrm{refl}(a + b); \\ \text{+-is-assoc}\ a\ b\ S(c) &:= \mathrm{ap}_S(\text{+-is-assoc}\ a\ b\ c). \end{aligned}$$

Other Inductive Types

The empty and unit types (logical false and true):

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In HoTT, *higher* inductive types can be defined, which can include path constructors as well as the point constructors that we have seen so far.

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Its corresponding induction principle is known as *path induction*: given a type A, there is a canonical term

$$\begin{aligned}\mathrm{J} : (C : ((x, y : A) \to x = y \to \mathrm{Type})) \to C\,x\,x\,\mathrm{refl}(x) \to \\ \to (x, y : A) \to (p : x = y) \to C\,x\,y\,p\end{aligned}$$

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There is also a version of this principle, which holds an endpoint fixed, *based path induction*: given a type A and a term $a : A$, there is a canonical term

$$\begin{aligned} &\mathrm{J}' : (C : ((x : A) \to a = x \to \mathrm{Type})) \to C\,x\,\mathrm{refl}(x) \to \\ &\quad \to (x : A) \to (p : a = x) \to C\,x\,p \end{aligned}$$

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Being a 0-type (a type with h-level 0) means being contractible, which is expressed as:

$$\begin{aligned} \text{is-contr} &: \text{Type} \to \text{Type}, \\ \text{is-contr}(A) &:= (x : A) \times (y : A) \to x = y. \end{aligned}$$

H-Levels

So far, only vanilla MLTT has been covered.
The h-level of a type is the number of times that $=$ must be iterated before it trivializes.
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Rest assured that the identity type of a contractible type is itself contractible.

More H-Levels

Being a *proposition* means being a 1-type (having “at most one” term/proof):

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It can be proven (with some effort) that $\mathbb{N}$ is a set (this follows by Hedberg’s Theorem, or from the “encode-decode” method).

Equivalence

Given types A, B, being an equivalence is a property of functions $A \to B$:

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This is actually different from the naïve translation of “is bijective” (since there might be different *proofs* that $f(x) = y$).
We can define the type of all equivalences between two types:

$$A \simeq B := (f : A \to B) \times \text{is-equiv}(f),$$

which is a “nice” definition since is-equiv(f) is always a proposition.

Univalence

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- Sets with the same cardinality are equal (don't confuse *sets* with *subsets*).
- Isomorphic groups are equal (don't confuse *groups* with *subgroups*).

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It's already used informally, for convenience. If G and H are isomorphic groups, and G is sqaunchy, then you can bet that H is squanchy. HoTT provides a rigorous justification for this.

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$$\mathbb{M} : \text{Type}, \qquad \rotatebox{90}{1} : \mathbb{M}, \qquad \rotatebox{90}{S} : \mathbb{M} \to \mathbb{M},$$

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By characterizing the identity type, univalence lets us prove that $\mathrm{Set} :\equiv (A : \mathrm{Type}) \times \text{is-set}(A)$ is a 3-type (i.e., its identity types are sets). Given sets A, B, the type $A = B$ is equivalent (and hence equal!) to the *set* of bijections from A to B.

Higher Inductive Types

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This features the *higher constructor* squash.
Intuitively, to define a function out of $\|A\|$, you may use a term of A, provided that the value you are defining does not depend on which term you select (for example, the codomain type could be a proposition).

More Logic

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The axiom of choice: for any *set* A and any family of *sets* $B : A \to \mathrm{Type}$, we have

$$((x : A) \to \|B(x)\|) \to \|(x : A) \to B(x)\|.$$

More HITs

Quotient Types

Given a relation $R : A \to A \to \mathrm{Type}$, you can form the quotient of A by R as a HIT:

$$[-] : A \to A/R, \qquad \text{squash} : (x\,y : A) \to R(x,y) \to [x] = [y],$$
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Defined by

$$\mathbb{S}^1 : \text{Type}, \qquad \text{base} : \mathbb{S}^1, \qquad \text{loop} : \text{base} = \text{base},$$

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Proof Assistants

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- Ensures that you don't make mistakes when things get complicated.
- Turns mathematics into a fun computer game.

Cubical Type Theory

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Talk to me about this later...

Further Reading

Introduction to HoTT:
https://arxiv.org/abs/2212.11082

The HoTT Book:
https://homotopytypetheory.org/book/

The 1Lab:
https://1lab.dev/

The HoTT Game:
https://thehottgameguide.readthedocs.io/en/latest/index.html