Finding factors in Berlekamp's Algebra

UQ Mathematics Student Society

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The Problem

Suppose p is prime. Let $\mathbb{Z}_p$ be the field of integers mod p.

Take a polynomial $f \in \mathbb{Z}_p[x]$.

Suppose, for simplicity, that f is squarefree[1].

How do we find the factors of f?

[1] has no repeated factors. The map $f \mapsto f/\gcd(f, f')$ deletes repeated factors.

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Problem Motivation

Ideas?

Let's get the obvious out of the way.

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- this is boring (and slow)

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Observation 1 We don't have to find every factor in one go.

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If we can reliably produce even **just one** non-trivial divisor of f, then repeated application of our procedure will suffice to find every factor.

Non-Trivial Divisors

Suppose f has factors f_1, f_2, …, f_n

A non-trivial divisor g of f must contain at least one factor f_i of f

We might say that g splits f in two.

$$f = (f_1 f_2 f_3)(f_4 \dots f_n)$$

$$g = (f_1 f_2 f_3) \qquad f/g = (f_4 \dots f_n)$$

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Non-Trivial Divisors

If we can come up with a `findNonTrivialDivisor` function, our factoring algorithm might look something like this:

```haskell
factor :: Polynomial -> Set Polynomial
factor f = case findNonTrivialDivisor f of
    Just g ->
        let h = f / g in
        Set.union ( factor g ) ( factor h )
    Nothing ->
        -- f is irreducible
        Set.singleton f
```

Non-Trivial Divisors

We want to find $g \in \mathbb{Z}_p[x]$ such that:

1. At least one factor of f divides g
2. At least one factor of f doesn't divide g
3. g divides f

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the map $g \mapsto \gcd(f, g)$ deletes the factors of g that don't divide f!

Non-Trivial Divisors

We want to find $g \in \mathbb{Z}_p[x]$ such that:

1. At least one factor of f divides g
2. At least one factor of f doesn't divide g
3. ~~All the factors of g divide f~~ Apply $g \mapsto \gcd(f, g)$!

the map $g \mapsto \gcd(f, g)$ deletes the factors of g that don't divide f!

Non-Trivial Divisors

Where might we find such a polynomial g?

It's kinda stupid to look for g in all of $\mathbb{Z}_p[x]$

The majority of these polynomials have degree greater than that of f, and so do not divide it.

Instead, we should search only the set of polynomials whose degree is less than that of f

$$D_f \ = \ \{ \, h \in \mathbb{Z}_p[x] \ : \ \deg(h) < \deg(f) \, \}$$

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Non-Trivial Divisors (Recap)

We've reduced our problem, finding all the factors of f, to the following problem:

Find any $g \in D_f$ such that:

1. At least one factor of f divides g
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We will call such polynomials **based**

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Observation 2 This feels like a quotient ring situation.

Consider the ring of polynomials $\bmod f$

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Note. this is basically D_f but closed under multiplication.

The Quotient Ring

Does our new multiplication $(\cdot)$ on A_f preserve the properties we care about?

If g and h are based, is $g \cdot h$ based?

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The Quotient Ring

Suppose g is based

wlog write $\gcd(f, g) = f_1 \cdots f_k$

Now $h = f / \gcd(f, g) = f_{k+1} \cdots f_n$ is also based

Notice that every factor of f divides $g \cdot h$

So $g \cdot h$ isn't based

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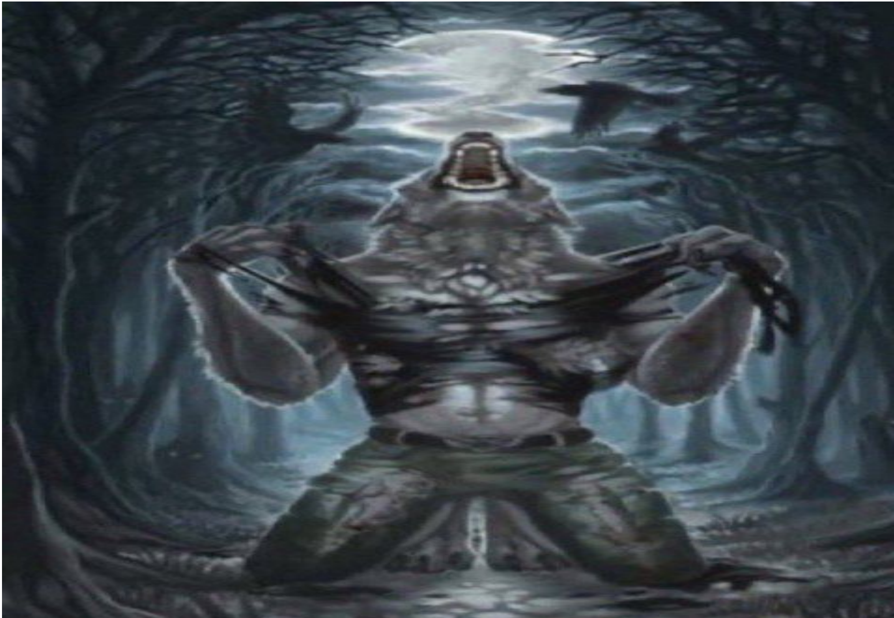
frick



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The Quotient Ring

Theorem (Chinese remainder)

Suppose R is a commutative ring, and $I_1, I_2, \ldots, I_n$ are ideals of R such that for every pair i, j of non-equal indices

$$R = \{ ax + by : a \in I_i, b \in I_j, x, y \in R \}$$

Then, the function

$$\sigma : R/\bigcap_{i=1}^{n} I_i \to R/I_1 \times R/I_2 \times \cdots \times R/I_n$$
$$[r] \mapsto ([r], [r], \ldots, [r])$$

is a ring isomorphism.

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The Quotient Ring

Notice that

$$\langle f \rangle = \langle f_1 \rangle \cap \langle f_2 \rangle \cap \cdots \cap \langle f_n \rangle$$

Moreover, it follows from the Chinese remainder theorem that there is an isomorphism

$$\sigma : A_f \to \mathbb{Z}_p[x]/\langle f_1 \rangle \times \mathbb{Z}_p[x]/\langle f_2 \rangle \times \cdots \times \mathbb{Z}_p[x]/\langle f_n \rangle$$

The Quotient Ring

The map

$$\sigma \ : \ g \ \mapsto \ (\ g \bmod f_1, \ g \bmod f_2, \ \ldots, \ g \bmod f_n \ )$$

is an isomorphism

The Quotient Ring

Now it's obvious that if g and h are based, then $g \cdot h$ is either based or zero.

$$\begin{aligned} \sigma(g \cdot h) &= \sigma(g) \cdot \sigma(h) \\ &= (\, 0, *, \ldots, * \,) \cdot (\, *, 0, \ldots, 0 \,) \\ &= (\, 0 \cdot *, * \cdot 0, \ldots, * \cdot 0 \,) \\ &= (\, 0, 0, \ldots, 0 \,) \end{aligned}$$

Moreover, if $g \cdot h$ is based or zero then at least one of g or h is based or zero.

- if $\sigma(g)$ is zero in the i^{th} component, then so is $\sigma(g \cdot h)$

The Quotient Ring

Notice that if g is based, then g^k is based for every positive k.

$$\sigma(g^k) = \sigma(g)^k = (\,*, \cdots, 0, \cdots, *\,)^k = (\,*, \cdots, 0, \cdots, *\,)$$

The set of based polynomials is closed under exponentiation.

The Quotient Ring

Give a name to the exponentiation map:

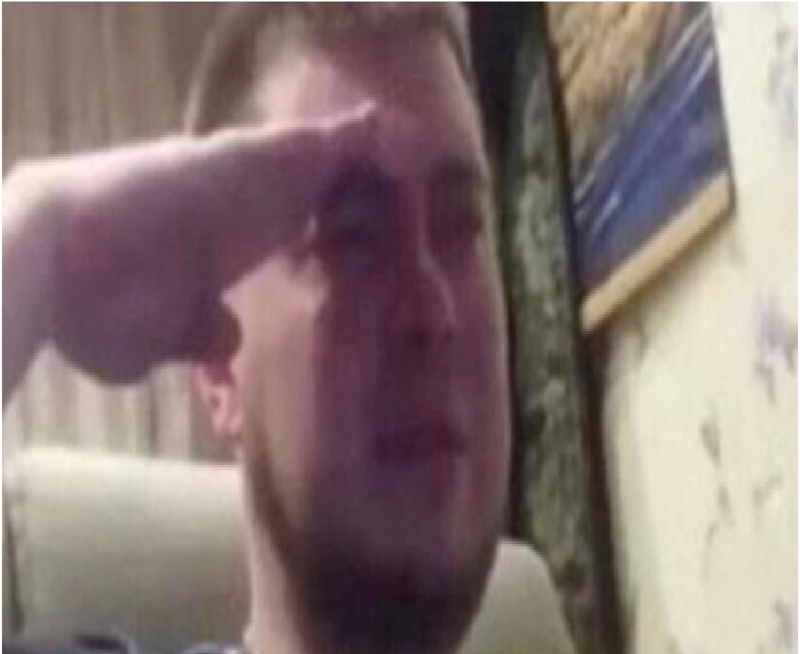
$$\begin{aligned} Q_k : A_f &\to A_f \\ g &\mapsto g^k \end{aligned}$$

The Quotient Ring

At this point we must make a sacrifice.

We must abandon some based polynomials.

frick



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We're going to pick some k and look at the subset in A_f that is fixed by Q_k

- which k should we pick?
- what properties do we want?

The Quotient Ring

Suppose g and h are in fix(Q_k)

$$Q_k(g \cdot h) = (g \cdot h)^k = g^k \cdot h^k = g \cdot h$$

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Suppose g and h are in fix(Q_k)

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- set $k = p$ (the same p as in $\mathbb{Z}_p[x]$)

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$$B_f \stackrel{\mathsf{def}}{=} \mathsf{fix}(Q_p)$$

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B_f is called the Berlekamp subalgebra.

Berlekamp's Algebra

Some facts:

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1. $\mathbb{Z}_p \subset B_f \subset A_f \subset \mathbb{Z}_p[x]$

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3. $B_f \; = \; \mathsf{fix}(Q_p) \; = \{ \, x \in A_f \; : \; x^p - x = 0 \, \} = \ker(Q_p - \mathsf{id})$

Berlekamp's Algebra

We can encode $Q_f - \mathrm{id}$ as an A_f valued matrix.

[2] other methods are available

Berlekamp's Algebra

We can encode $Q_f - \mathrm{id}$ as an A_f valued matrix.

Then we can use Gaussian elimination[2] to produce a basis for its nullspace.

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Berlekamp's Algebra

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Then we can use Gaussian elimination[2] to produce a basis for its nullspace.

The members of B_f are precisely the linear combinations of the elements of this basis!

[2] other methods are available

Berlekamp's Algebra

Lock in guys.

You ain't ready for this.

Berlekamp's Algebra

Have you seen this equality? For every $h \in \mathbb{Z}_p[x]$

$$\prod_{c \in \mathbb{Z}_p} (h + c) = h^p - h$$

proof on joelrichardson.au if u want

Berlekamp's Algebra

For every $h \in B_f$

$$\prod_{c \in \mathbb{Z}_p} (h + c) = 0 \mod f$$

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For every $h \in B_f$

$$\prod_{c \in \mathbb{Z}_p} (h + c) = 0 \mod f$$

- the product of the $(h + c)$s is based or zero
- at least one $h + c$ is based!
- at least one $\gcd(f, h + c)$ is a non-trivial divisor of f!

Berlekamp's Algebra

Recall our earlier code block:

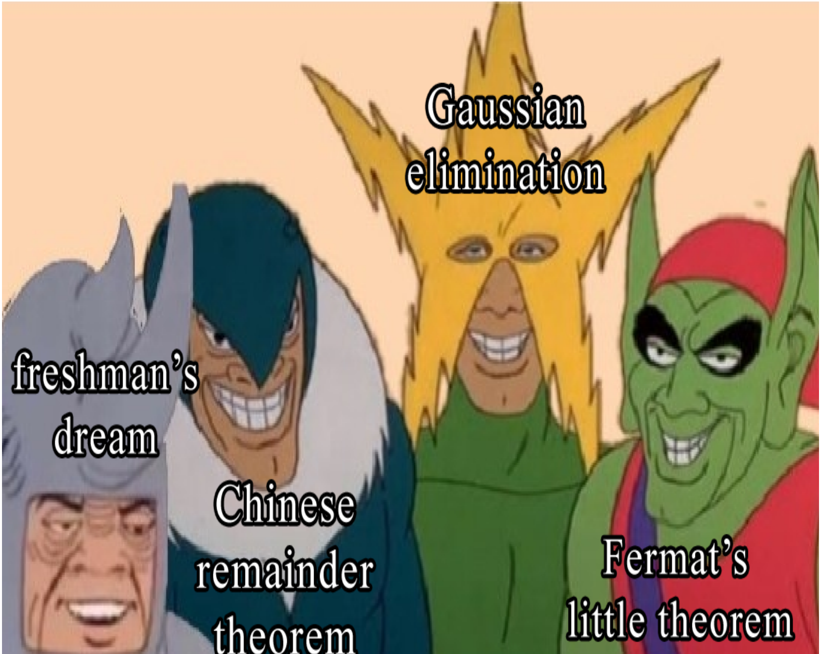
```
factor :: Polynomial -> Set Polynomial
factor f = case findNonTrivialDivisor f of
    Just g ->
        let h = f / g in
        Set.union ( factor g ) ( factor h )
    Nothing ->
        -- f is irreducible
        Set.singleton f
```

Berlekamp's Algebra

We can now implement `findNonTrivialDivisor`

```
findNonTrivialDivisor :: Polynomial -> Maybe Polynomial
findNonTrivialDivisor f = case nullspaceBasis (berlekampMatrix f) of
    basis | length basis < 2 ->
        Nothing
    basis ->
        let h = head basis in
        find ( isNonZeroNonUnit ) [ gcd f (h + c) | c <- field ]
        -- dont forget to apply the ^^^ gcd we talked about earlier!
```

Any Questions???



more.

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So,

$$f \;=\; \gcd\left(f, \prod_{c \in \mathbb{Z}_p} h + c\right)$$

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We know that the following is a multiple of f

$$\prod_{c \in \mathbb{Z}_p} h + c$$

So,

$$f \;=\; \gcd\left(f,\, \prod_{c \in \mathbb{Z}_p} h + c\right)$$

Pretty easy to show that

$$f \;=\; \prod_{c \in \mathbb{Z}_p} \gcd\left(f,\, h + c\right)$$

Berlekamp's Algorithm

```
factorBerlekamp :: Polynomial -> Set Polynomial
factorBerlekamp f = case nullspaceBasis (berlekampMatrix f) of
    basis | length basis > 1 ->
        let h = head basis in -- element of B_f
        let terms = filter
                ( isNonZeroNonUnit )
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        in Set.unionMap factorBerlekamp terms
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funny: the dimension of B_f is the number of factors of f.