Mercator projection

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Question

What is the best method for constructing a flat map of the spherical Earth?

Regular Surfaces

$f : A \subseteq \mathbb{R}^m \to B \subseteq \mathbb{R}^n$ is a diffeo. if f and f^{-1} are smooth

A and B are diffeomorphic if there's a diffeo. between them

A set $S \subset \mathbb{R}^3$ is a regular surface if

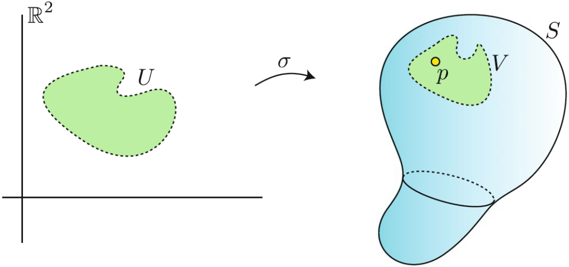


Figure: A Regular Surface (Tapp)

Examples

- Sphere S^2 and ellipsoid E
- Plane $\mathbb{R}^2$ and paraboloid $z = x^2 + y^2$
- Graph of smooth $f : U \to \mathbb{R}$ and domain U

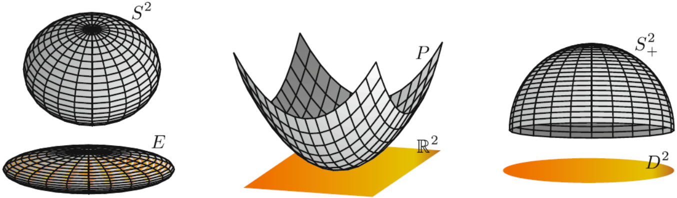


Figure: Diffeomorphisms (Tapp)

Tangent Plane

$$T_pS = \{\gamma'(0) \,:\, \gamma \text{ is a regular curve in } S \text{ and } \gamma(0) = p\}$$

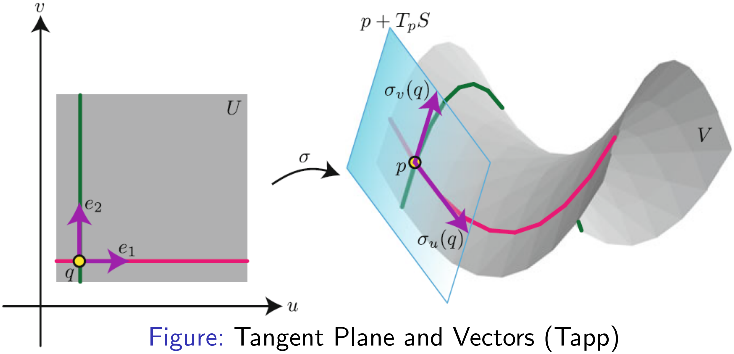


Figure: Tangent Plane and Vectors (Tapp)

Let $\sigma : U \subseteq \mathbb{R}^2 \to V \subseteq S$ be a diffeo. with $p \in V$

$$T_pS = \text{span}\{\sigma_x(p), \sigma_y(p)\}$$

Differential

Smooth $f : A \to B$ the differential $df_p : T_pA \to T_{f(p)}B$

$$df_p(v) = (f \circ \gamma)'(0)$$

γ is a regular curve s.t. $\gamma(0) = p$ and $\gamma'(0) = v$.

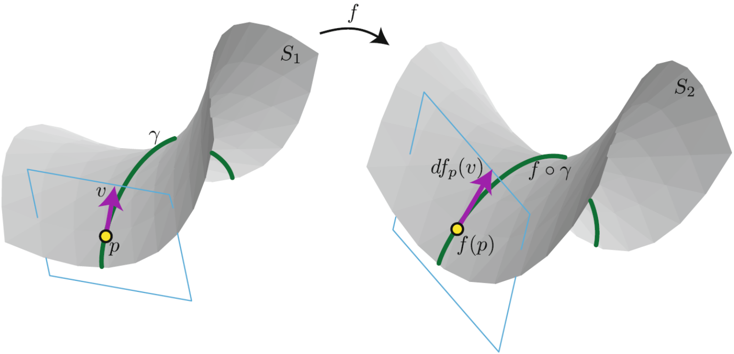


Figure: Differential (Tapp)

Equiareal and Conformal Maps

Let A, B be regular surfaces and let $f : A \to B$ diffeo. f is called an isometry if it preserves the inner product:

$$\langle v, w \rangle = \langle df_p(v), df_p(w) \rangle,$$

equiareal if it preserves area:

$$\|df_p\| := \frac{|df_p(v) \times df_p(w)|}{|v \times w|} = 1,$$

conformal if it preserves angles:

$$\angle(v, w) = \angle(df_p(v), df_p(w))$$

(for all $p \in A$ and for all $v, w \in T_pA$).

Archimedes

Consider the map $f : S^2 \setminus \{N, S\} \to C \subseteq \mathbb{R}^3$ given by

$$f(x, y, z) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, z \right)$$

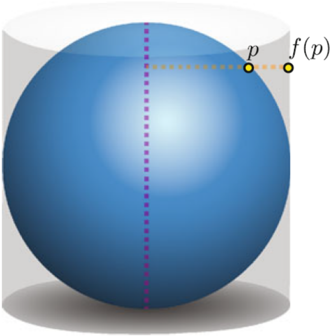


Figure: Archemides

Archimedes

Claim

Archimedes' map is equiareal

Proof.

Spherical coordinates $\sigma : (0,\pi) \times (0, 2\pi) \to S^2$

$$\begin{aligned} \sigma(u,v) &= (\sin u \cos v, \sin u \sin v, \cos u) \\ (f \circ \sigma)(u,v) &= (\cos v, \sin v, \cos u) \end{aligned}$$

Calculate $\|df_p\|$ using σ_u and σ_v.

$$\begin{aligned} df_p(\sigma_u) &= (f \circ \sigma)_u = (0, 0, -\sin v) \\ df_p(\sigma_v) &= (f \circ \sigma)_v = (-\sin u, \cos u, 0) \end{aligned}$$

$$\|df_p\| = \frac{|df_p(\sigma_u) \times df_p(\sigma_v)|}{|\sigma_u \times \sigma_v|} = \frac{|\sin v|}{|\sin v|} = 1$$

□

Archimedes Map

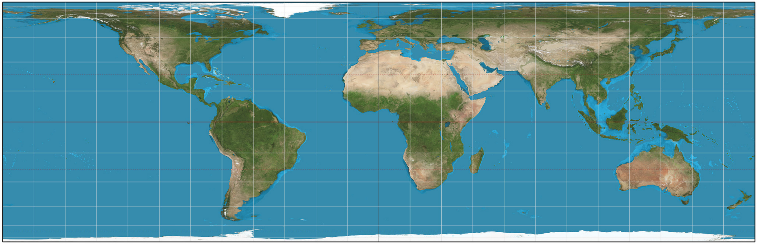


Figure: Archemides Map

Stereographic Projection

Consider $\Phi : S^2 \setminus \{N\} \to \mathbb{R}^2$ given by $\Phi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$

Claim

The stereographic projection Φ is conformal

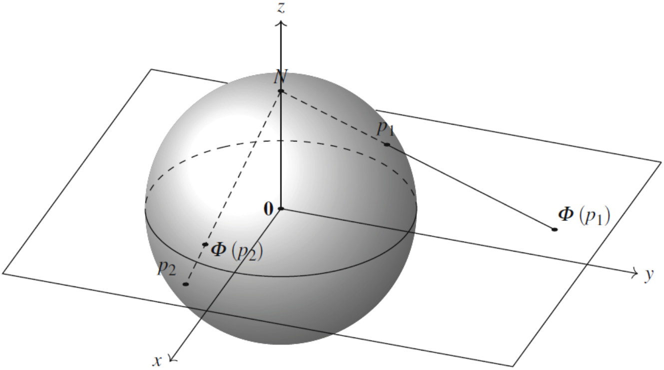


Figure: Stereographic Projection (Nam-Hoon)

Stereographic Projection Conformal 1

Let $p \in S^2 \setminus \{N\}$, $v_1, v_2 \in T_pS^2$, $\theta = \angle(v_1, v_2)$.

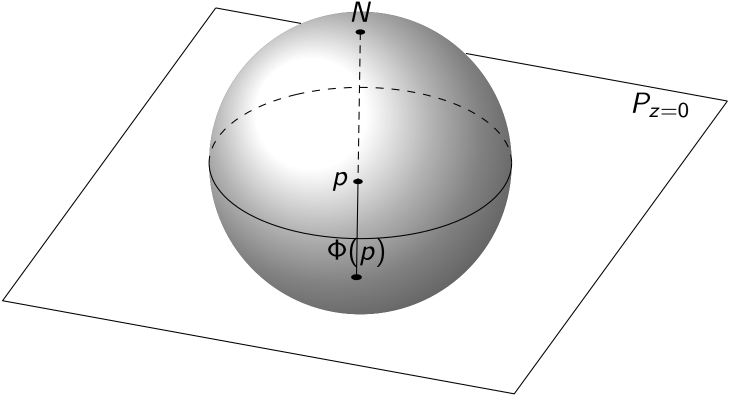


Figure: Stereographic Projection Conformal 1 (Nam-Hoon)

Stereographic Projection Conformal 2

Planes $P_1, P_2 \subseteq S^2$ through p, N tangent to v_1, v_2 resp.
Let $C_1 = S^2 \cap P_1$ and $C_2 = S^2 \cap P_2$.
Note that $\theta = \angle_p(C_1, C_2) = \angle_N(C_1, C_2)$

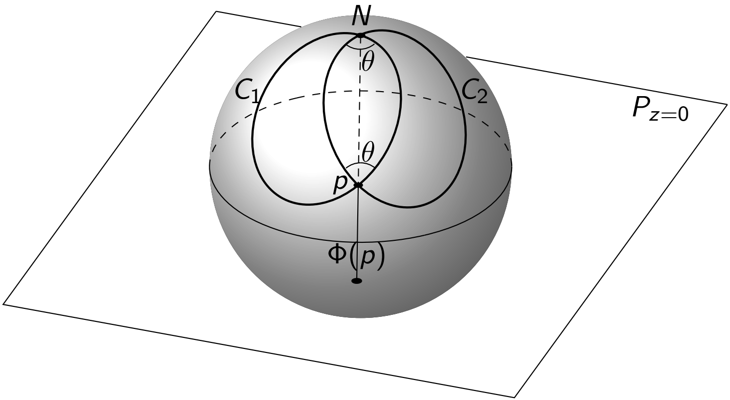


Figure: Stereographic Projection Conformal 2 (Nam-Hoon)

Stereographic Projection Conformal 3

Let $L_1 = \Phi(C_1) = P_1 \cap P_{z=0}$ and $L_2 = \Phi(C_2) = P_2 \cap P_{z=0}$.

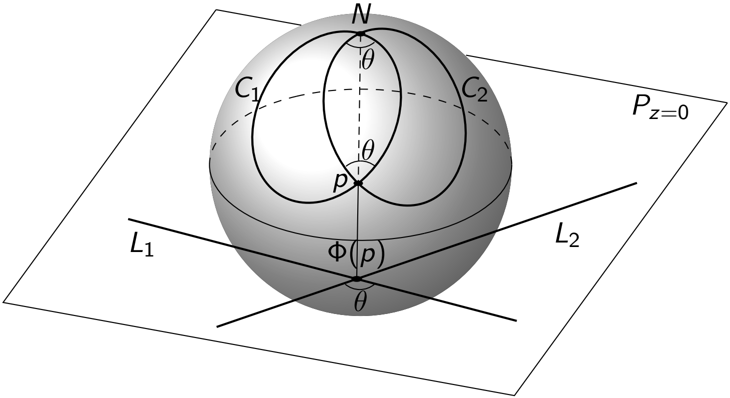


Figure: Stereographic Projection Conformal 3 (Nam-Hoon)

Stereographic Projection Conformal 4

Let $L_1' = P_1 \cap P_{z=1}$ and $L_2' = P_2 \cap P_{z=1}$.

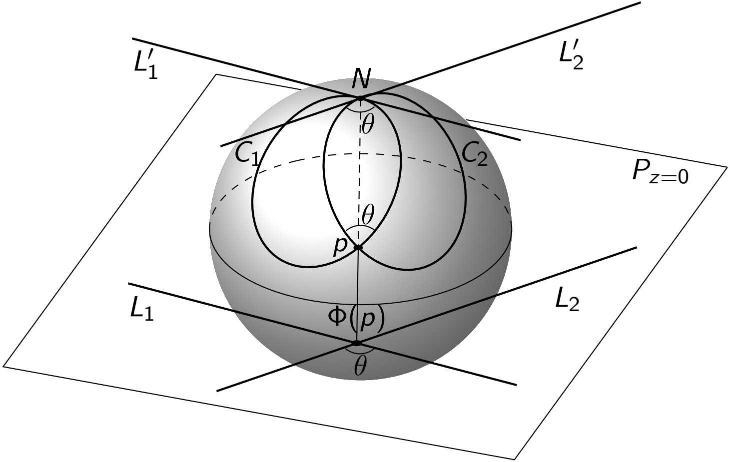


Figure: Stereographic Projection Conformal 4 (Nam-Hoon)

Stereographic Projection Conformal 5

Because $P_{z=0} \parallel P_{z=1}$, we have $L_1 \parallel L_1'$ and $L_2 \parallel L_2'$

$$\angle_{\Phi(p)}(L_1, L_2) = \angle_N(L_1', L_2') = \angle_N(C_1, C_2) = \theta$$

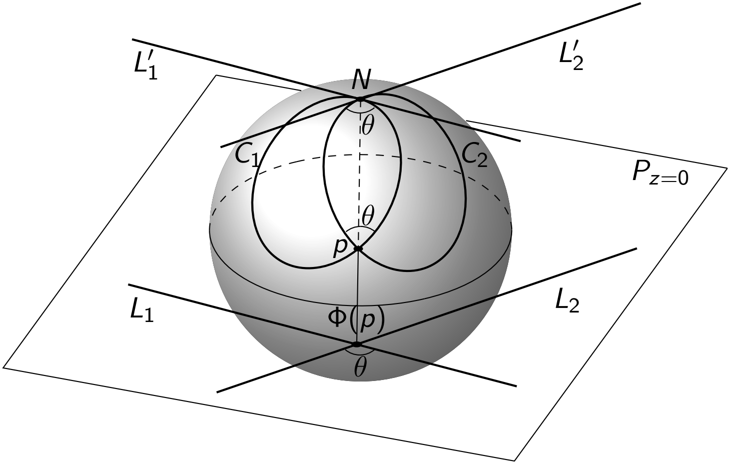


Figure: Stereographic Projection Conformal 5 (Nam-Hoon)

Stereographic Projection Map

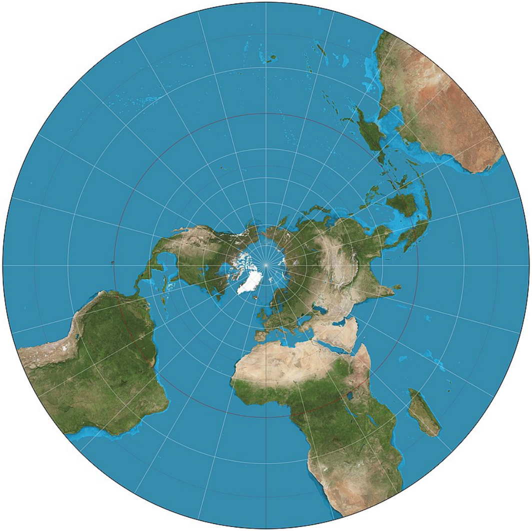


Figure: Stereographic Projection Map

Mercator Projection

Goal: modify the spherical coordinate chart to make it conformal.
$\sigma : (0, \pi) \times (0, 2\pi) \to S^2$

$$\sigma(u, v) = (\sin u \cos v, \sin u \sin v, \cos u)$$

Fact: $f : A \to B$ is conformal $\iff$ for all $p \in A$ there are $v, w \in T_p A$ such that

$$v \perp w, \quad df_p(v) \perp df_p(w) \text{ and } \frac{|df_p(v)|}{|v|}, \frac{|df_p(w)|}{|w|}.$$

Observe $\sigma_u \perp \sigma_v$ but $|\sigma_u| = 1$ and $|\sigma_v| = \sin u$. Want $|\sigma_v| = 1$
Compose with $f : (a, b) \times (0, 2\pi) \to (0, \pi) \times (0, 2\pi)$

$$f(t, v) = (\phi(t), v)$$

$$|(\sigma \circ f)_t| = |\phi'(t)|, \quad |(\sigma \circ f)_v| = \sin(\phi(t)).$$

Mercator Projection

ODE

$$\phi'(t) = \sin(\phi(t))$$

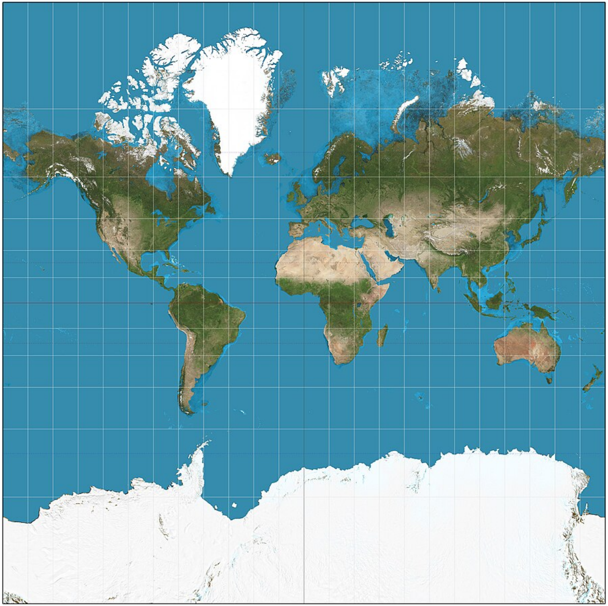
Solution

$$\phi(t) = 2\cot^{-1}(e^{-t})$$

over $(-\infty, \infty)$ where $\phi(0) = \frac{\pi}{2}$.

Choose finite domain $(-L, L) \times (0, 2\pi)$ nbhds of poles are uncharted

Mercator Projection



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