Guaranteed performance under uncertainty

A quick overview of robust control

The title is an open problem!

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The University of Queensland

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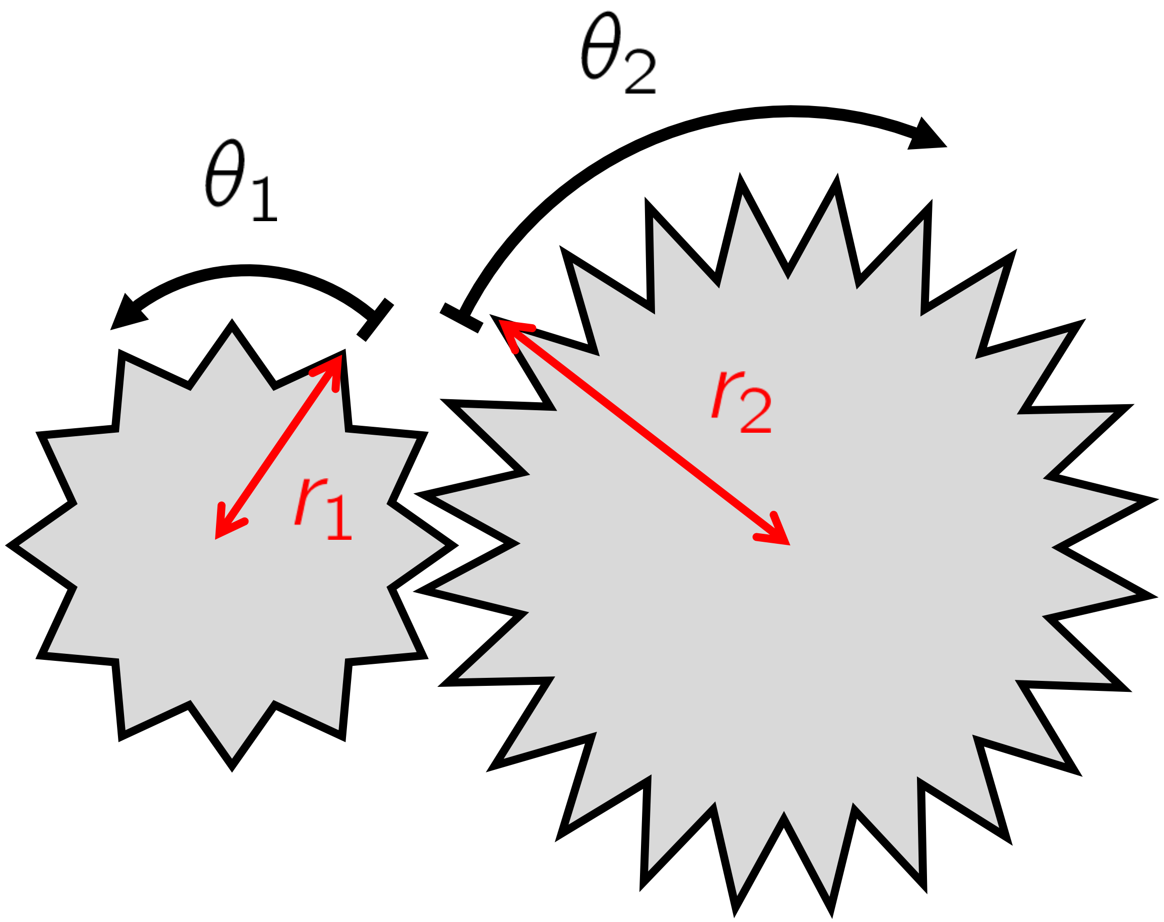
Contents

- Fundamentals and a taste of what's out there
- "Assumed":
  - MATH1051 – calculus, polynomials
  - MATH1052 – linear ODEs
  - Complex numbers
- Highly broad field!

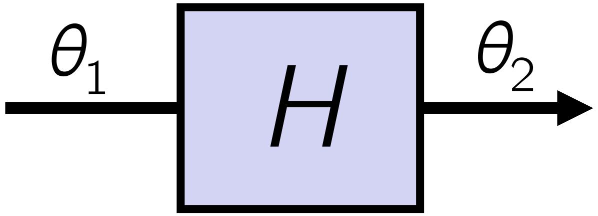
Gears



$$r_1\theta_1 = r_2\theta_2$$

$$\theta_2 = \frac{r_1}{r_2}\theta_1$$

$$= H\theta_1$$



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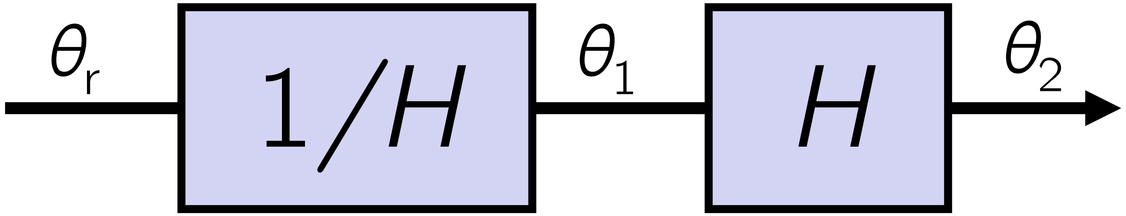
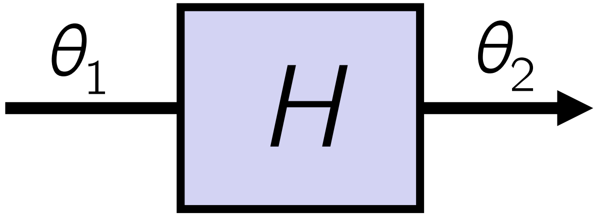
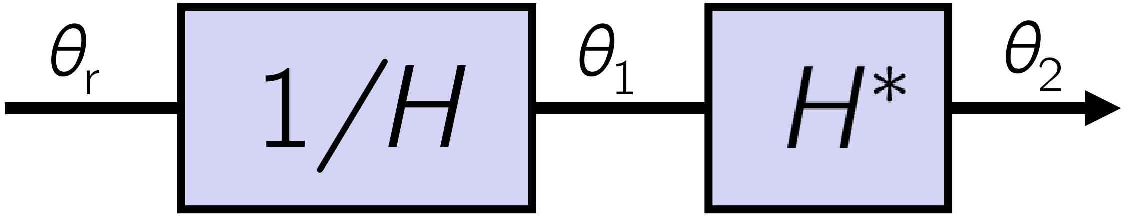


Choose θ_1 so that $\theta_2 = \theta_r$.

Set $\theta_1 = \frac{1}{H}\theta_r$.

Gears





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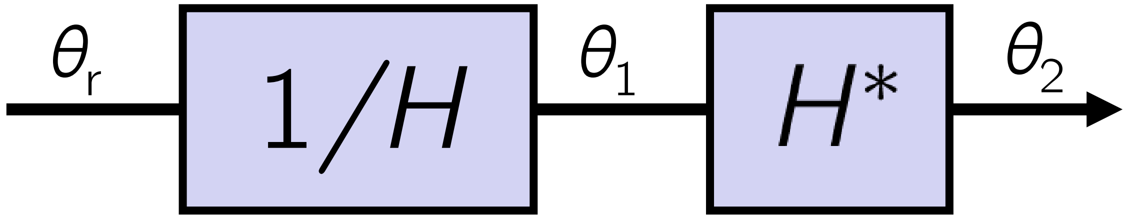
Choose θ_1 so that $\theta_2 = \theta_r$.

Set $\theta_1 = \frac{1}{H}\theta_r$.

Gears

Set $\theta_1 = \frac{1}{H}\theta_r$. What if $\theta_2 = H^*\theta_1$?



$$\theta_2 = H^*\theta_1 = \frac{H^*}{H}\theta_r$$

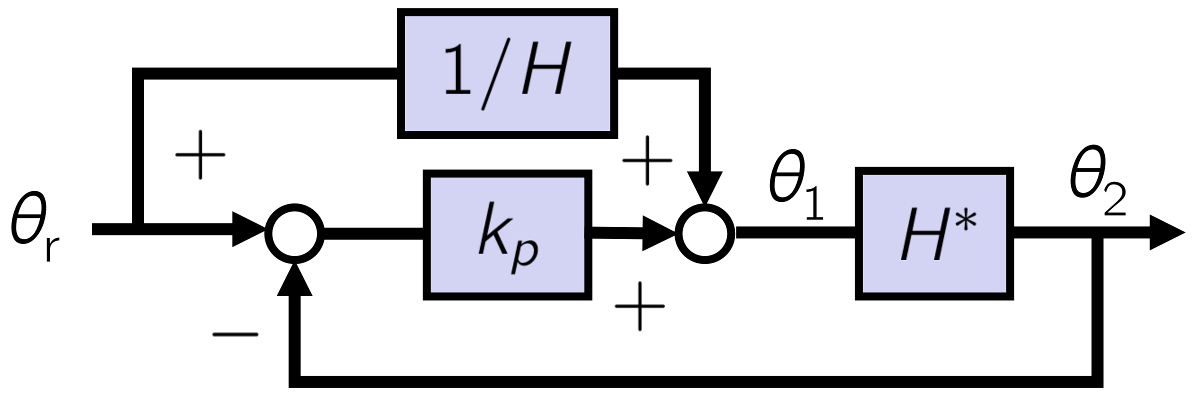
$\theta_2 = \theta_r$ only when $H^* = H$.

Need to account for error $\theta_r - \theta_2$.

Gears

Set $\theta_1 = \frac{1}{H}\theta_r + k_p(\theta_r - \theta_2)$. What if $\theta_2 = H^*\theta_1$?

"gain"



$$\theta_2 = H^*\theta_1 = \frac{H^*}{H}\theta_r + H^* k_p(\theta_r - \theta_2)$$

$$\theta_2 = \frac{\frac{H^*}{H} + H^* k_p}{1 + H^* k_p}\theta_r$$

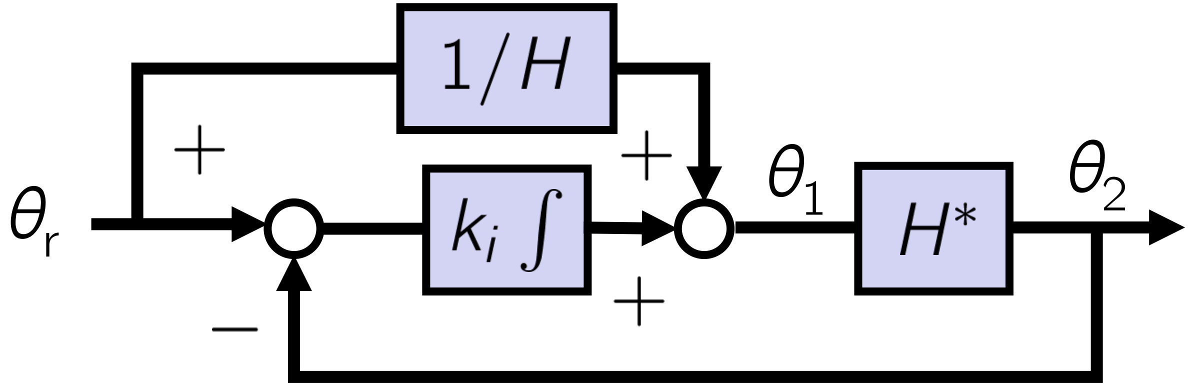
$$\frac{\theta_2}{\theta_r} = \frac{\frac{H^*}{H} + H^* k_p}{1 + H^* k_p}$$

$$\to 1 \text{ as } k_p \to \infty.$$

Gears

Set $\theta_1(t) = \frac{1}{H}\theta_r(t) + k_i \int_0^t (\theta_r(\tau) - \theta_2(\tau))\, \mathrm{d}\tau$.

What if $\theta_2(t) = H^*\theta_1(t)$?



Gears

Set $\theta_1(t) = \frac{1}{H}\theta_r(t) + k_i \int_0^t (\theta_r(\tau) - \theta_2(\tau))\, \mathrm{d}\tau$.

What if $\theta_2(t) = H^*\theta_1(t)$?

$$\theta_2(t) = H^* \left(\frac{1}{H}\theta_r(t) + k_i \int_0^t (\theta_r(\tau) - \theta_2(\tau))\, \mathrm{d}\tau \right)$$

$$\theta_2'(t) = H^* \left(\frac{1}{H}\theta_r'(t) + k_i(\theta_r(t) - \theta_2(t)) \right)$$

$$\theta_2'(t) + H^* k_i \theta_2(t) = H^* k_i \theta_r \xrightarrow{\theta_2(0)=0} \theta_2(t) = \theta_r(1 - e^{-H^* k_i t})$$

Gears

Set $\theta_1(t) = \frac{1}{H}\theta_r(t) + k_i \int_0^t (\theta_r(\tau) - \theta_2(\tau))\,\mathrm{d}\tau$.

What if $\theta_2(t) = H^*\theta_1(t)$?

Arbitrary convergence rate!

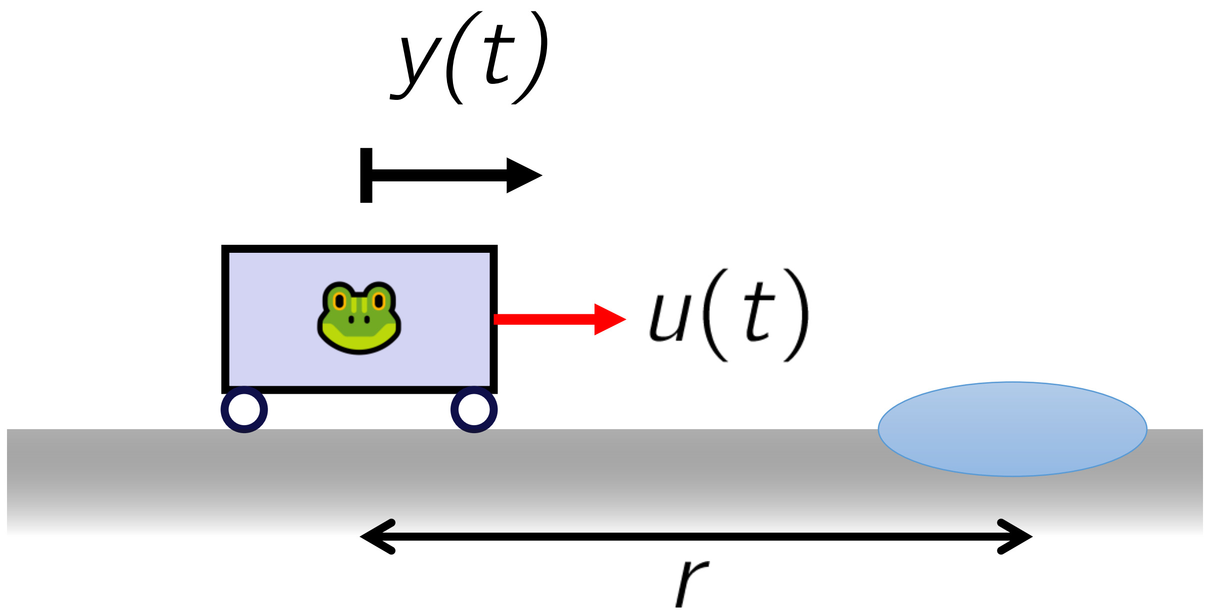
$$\theta_2(t) = \theta_r(1 - e^{-H^* k_i t})$$

$$\frac{\theta_2(t)}{\theta_r} \to 1 \text{ as } t \to \infty.$$

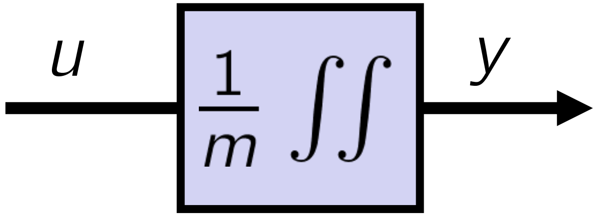
Independent of "prediction" $\frac{1}{H}\theta_r$!

Need feedback with accumulation of past errors.

Frog-cart



$$my''(t) = u(t)$$



“Invert” ??

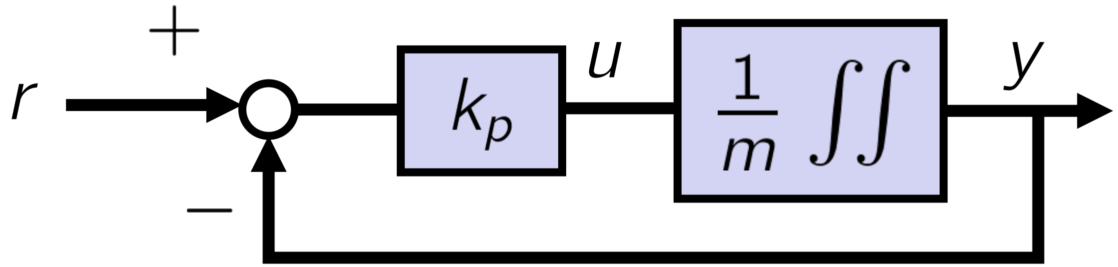
...could calculate how long to hold u constant, then how long to decelerate?

Same issues with inaccurate measurements.

Need u to be independent of m.

Frog-cart

Try $u(t) = k_p(r(t) - y(t))$. No explicit estimate of m.



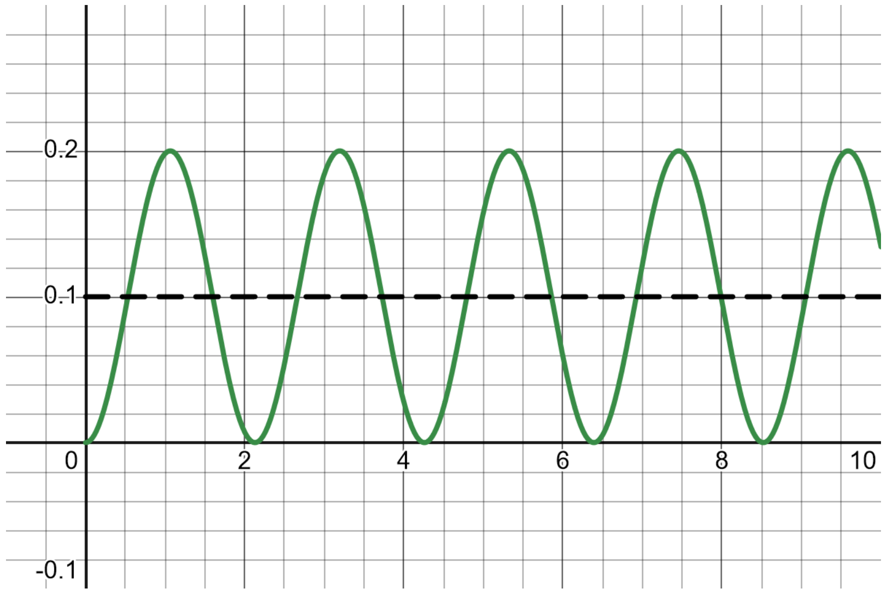
Frog-cart

Try $u(t) = k_p(r(t) - y(t))$. No explicit estimate of m.

$$my''(t) = u(t)$$
$$my''(t) = k_p r(t) - k_p y(t)$$
$$my''(t) + k_p y(t) = k_p r$$

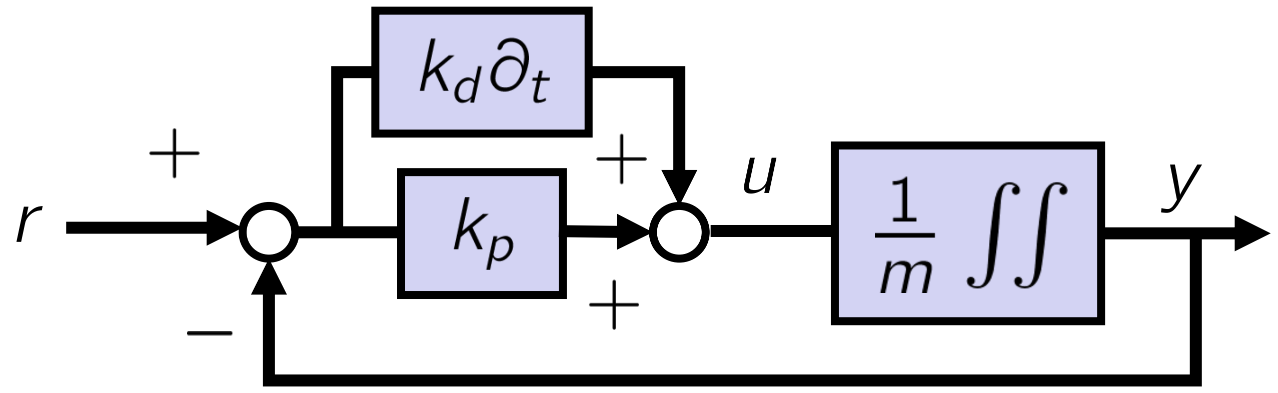
Prescribing $y(0) = y'(0) = 0$,

$$y(t) = r\left(1 - \cos\left(\sqrt{\frac{k_p}{m}}t\right)\right)$$



Frog-cart

Try $u(t) = k_p(r(t) - y(t)) + k_d \frac{\mathrm{d}}{\mathrm{d}t}(r(t) - y(t))$.



Frog-cart

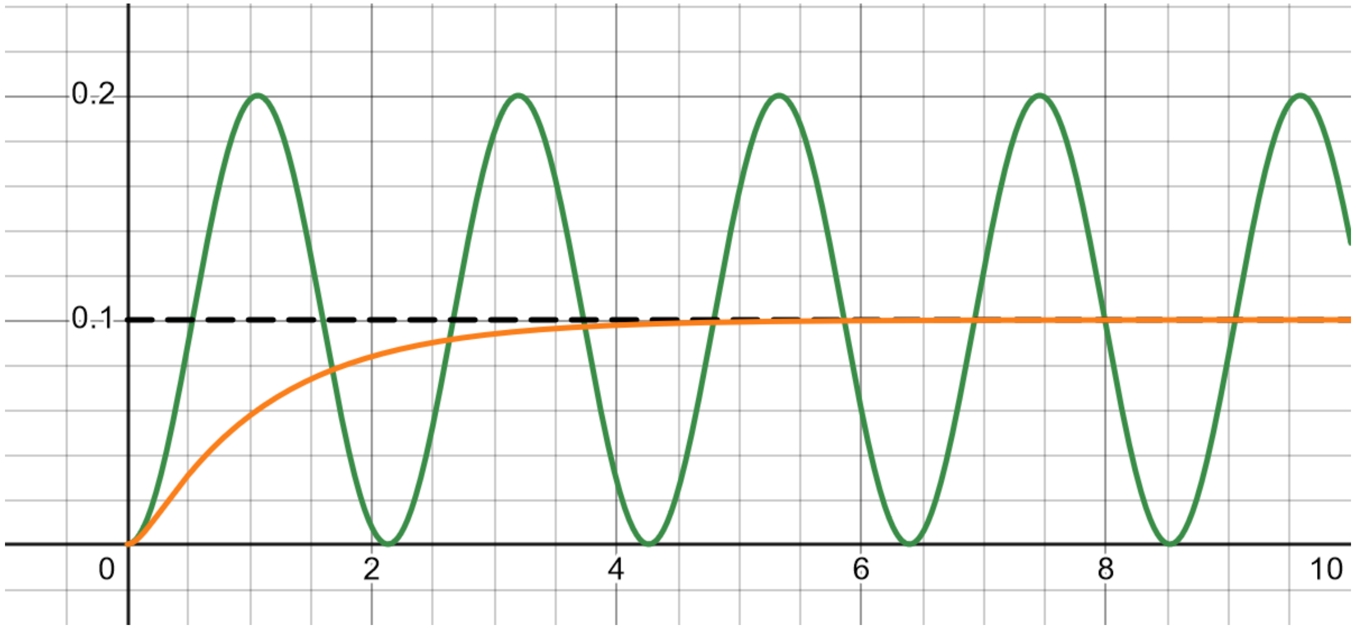
Try $u(t) = k_p(r(t) - y(t)) + k_d \frac{\mathrm{d}}{\mathrm{d}t}(r(t) - y(t))$.

$$
\begin{aligned}
my''(t) &= u(t) \\
my''(t) &= k_p r(t) - k_p y(t) + k_d r'(t) - k_d y'(t) \\
my''(t) + k_d y'(t) + k_p y(t) &= k_p r
\end{aligned}
$$

$$
y(t) = r\left[1 - \left(\frac{1}{2} - \frac{k_d}{2\sqrt{k_d^2 - 4k_p m}}\right)\exp\left(\frac{-k_d - \sqrt{k_d^2 - 4k_p m}}{2m}t\right) - \left(\frac{1}{2} + \frac{k_d}{2\sqrt{k_d^2 - 4k_p m}}\right)\exp\left(\frac{-k_d + \sqrt{k_d^2 - 4k_p m}}{2m}t\right)\right]
$$

Converges to 0 as long as exponents negative.

Frog-cart



$$y(t) = r\left[1 - \left(\frac{1}{2} - \frac{k_d}{2\sqrt{k_d^2 - 4k_p m}}\right)\exp\left(\frac{-k_d - \sqrt{k_d^2 - 4k_p m}}{2m}t\right) - \left(\frac{1}{2} + \frac{k_d}{2\sqrt{k_d^2 - 4k_p m}}\right)\exp\left(\frac{-k_d + \sqrt{k_d^2 - 4k_p m}}{2m}t\right)\right]$$

Converges to 0 as long as exponents negative.

A useful tool

Theorem

Every linear const-coeff ODE (*system*) is identified by an *impulse response* $h : [0, \infty) \to \mathbb{R}$ such that for all inputs $u : [0, \infty) \to \mathbb{R}$, the solution y with zero I.C. is

$$y(t) = (u * h)(t) := \int_0^t u(\tau)h(t - \tau)\,\mathrm{d}\tau.$$

(similar to Green's function, convolution kernel)

A useful tool

For $f : [0, \infty) \to \mathbb{C}$ satisfying some properties, the *Laplace transform* of f is

$$\mathcal{L}\{f\}(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,\mathrm{d}t,$$

defined for $s \in \mathbb{C}$ where the improper integral converges.

By convention, we write $\mathcal{L}\{f\} = F$.

The *inverse Laplace transform* has a difficult explicit form, but is often confirmed by direct calculation.

A useful tool

Theorem (Properties of $\mathcal{L}$)

1. $\mathcal{L}\{af + bg\} = aF + bG$
2. $\mathcal{L}\{f * g\} = FG$

Let h be the impulse response for some system. Then,

$$y = u * h$$
$$\mathcal{L}\{y\} = \mathcal{L}\{u * h\}$$
$$Y(s) = U(s)H(s)$$
$$H(s) = \frac{Y(s)}{U(s)}$$

The *transfer function* H completely describes the system, like h does.

Frog-cart

Theorem (Properties of $\mathcal{L}$)

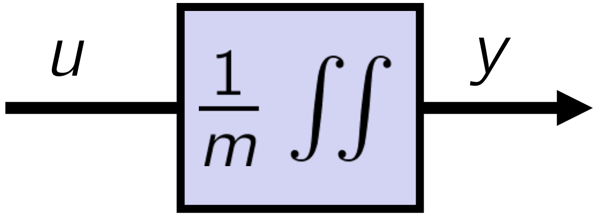
1. $\mathcal{L}\{af + bg\} = aF + bG$
2. $\mathcal{L}\{f * g\} = FG$
3. $\mathcal{L}\{f'\}(s) = sF(s) - f(0^-)$
4. $\mathcal{L}\{\int f\}(s) = \frac{F(s)}{s} + \frac{f'(0^-)}{s}$

$$my''(t) = u(t)$$

$$m(s^2Y(s) - sy(0) - y'(0)) = U(s)$$

$$Y(s) = \frac{1}{ms^2}U(s) + \frac{m(sy(0) + y'(0))}{ms^2}$$

$$H(s) := \frac{Y(s)}{U(s)} = \frac{1}{ms^2}$$

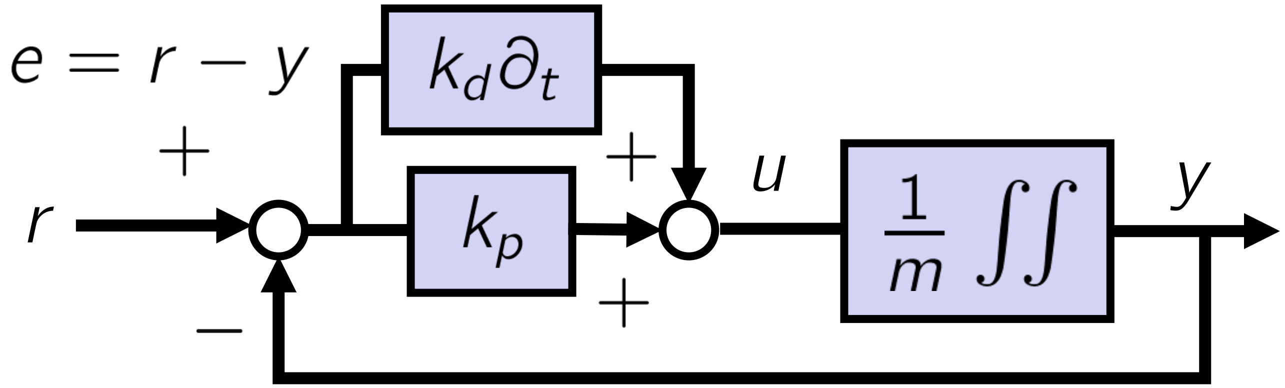


Frog-cart

$$u(t) = k_p(r(t) - y(t)) + k_d(r'(t) - y'(t))$$

$$U(s) = k_p(R(s) - Y(s)) + k_d s(R(s) - Y(s))$$

$$U(s) = \underbrace{(k_p + k_d s)}_{C(s)}\underbrace{(R(s) - Y(s))}_{E(s)}$$



Frog-cart

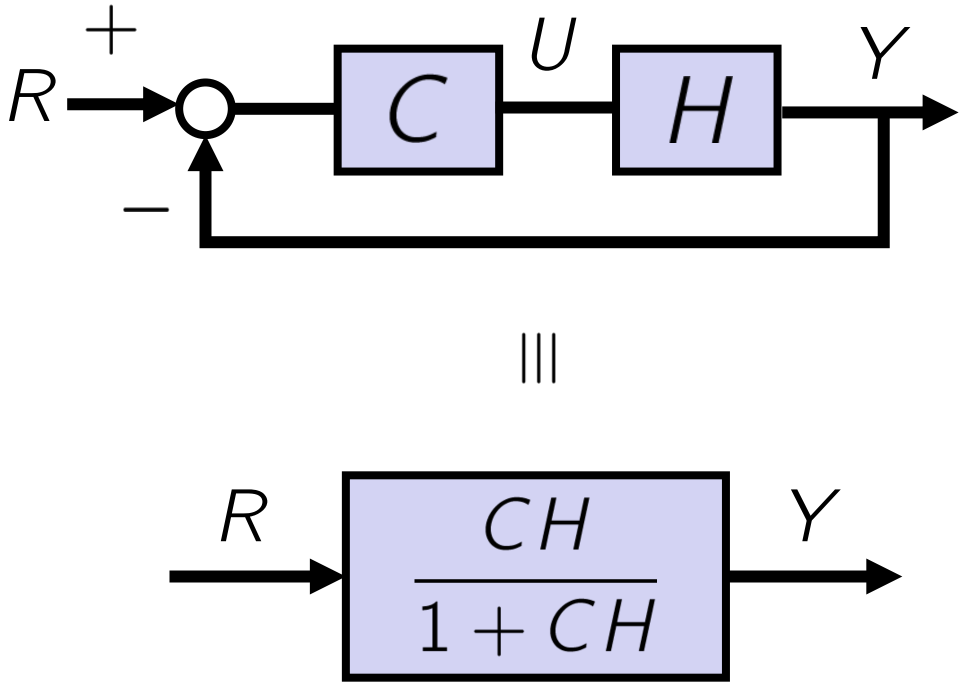
$$H(s) := \frac{Y(s)}{U(s)}$$

$$\implies Y = HU = HC(R - Y)$$

$$Y + CHY = CHR$$

$$\frac{Y}{R} = \frac{CH}{1 + CH}$$

Another transfer function, this time relating *reference* to output!



Frog-cart

$$Y = \frac{CH}{1+CH}R, \qquad H(s) := \frac{Y(s)}{U(s)} = \frac{1}{ms^2}, \qquad C(s) = k_p + k_d s$$

$$\frac{Y}{R} = \frac{\frac{k_p+k_d s}{ms^2}}{1+\frac{k_p+k_d s}{ms^2}} = \frac{k_d s + k_p}{ms^2 + k_d s + k_p} \qquad r(t) = r \xrightarrow{\mathcal{L}} R(s) = \frac{r}{s}$$

$$Y = \underbrace{\frac{k_d s + k_p}{ms^2 + k_d s + k_p}}_{m(s-p_1)(s-p_2)} \cdot \frac{r}{s} = r\left[\frac{A}{s} + \frac{B}{s-p_1} + \frac{C}{s-p_2}\right]$$

$$\mathcal{L}\{e^{zt}\} = \frac{1}{s-z}$$

$$y(t) = r\left[A + Be^{p_1 t} + Ce^{p_2 t}\right]$$

Frog-cart

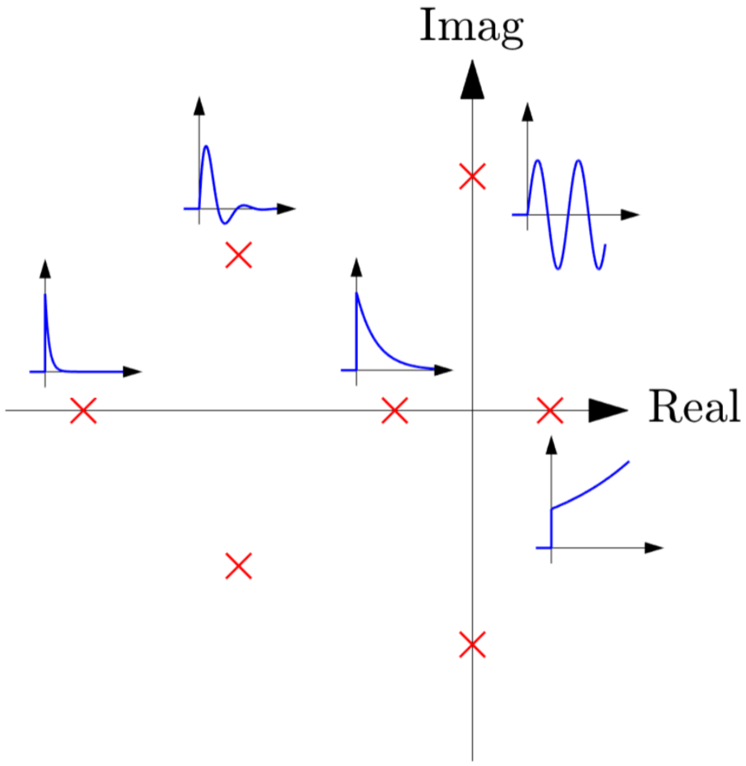
In general, with $m \leq n$ for causality*,

$$a_n y^{(n)}(t) + \cdots + a_0 y(t) = b_m u^{(m)}(t) + \cdots + b_0 u(t)$$

$$H(s) := \frac{Y(s)}{U(s)} = \frac{b_m s^m + \cdots + b_0}{a_n s^n + \cdots + a_0}$$

$$= \frac{\cdots}{(s - p_1)^{k_1}(s - p_2)^{k_2} \cdots}, \; p_i \in \mathbb{C}$$

$$\mathcal{L}\{e^{at} e^{ibt}\} = \frac{1}{s - (a + bi)}$$



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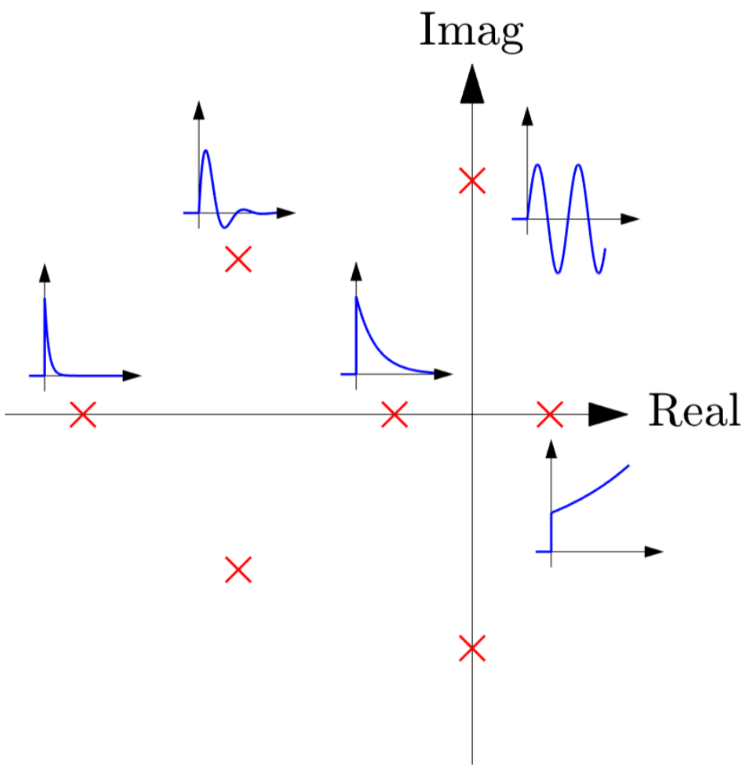


Same result applies to closed system,

$$\frac{CH}{1+CH} = \frac{\cdots}{(s-p_1)^{k_1}(s-p_2)^{k_2}\cdots}, \; p_i \in \mathbb{C}$$

A system (open or closed) is BIBO-stable iff $\mathrm{Re}(p_i) < 0$.

Frog-cart

$$y(t) = r\left[A + Be^{p_1 t} + Ce^{p_2 t}\right]$$

$$y(t) = r\left[1 - \left(\frac{1}{2} - \frac{k_d}{2\sqrt{k_d^2 - 4k_p m}}\right)\exp\left(\frac{-k_d - \sqrt{k_d^2 - 4k_p m}}{2m}t\right) - \left(\frac{1}{2} + \frac{k_d}{2\sqrt{k_d^2 - 4k_p m}}\right)\exp\left(\frac{-k_d + \sqrt{k_d^2 - 4k_p m}}{2m}t\right)\right]$$

Need $\mathrm{Re}(p_1), \mathrm{Re}(p_2) < 0$ for convergence (stability).

Need $A = 1$ for $y \to r$ as $t \to \infty$.

What is A?

Frog-cart

Theorem (Final Value)

If $Y(s)$ is stable, $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s)$.

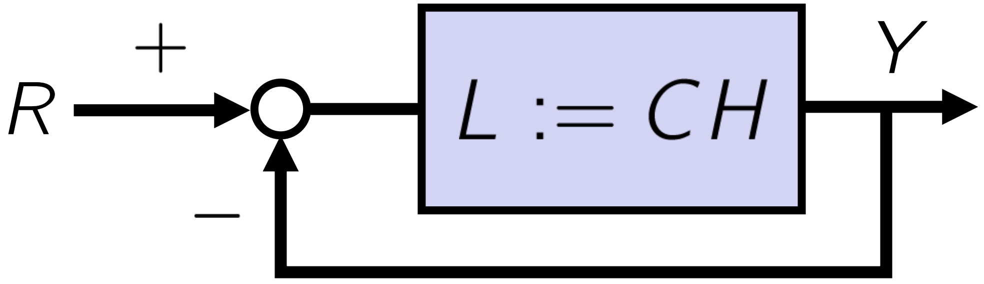
$$r(t) = r, \quad R(s) = r/s$$

$$\lim_{s\to 0} sY(s) = \lim_{s\to 0} sR(s)\frac{CH}{1+CH} = \lim_{s\to 0} s\frac{r}{s}\frac{CH}{1+CH}$$

$$= r\lim_{s\to 0}\frac{CH}{1+CH} \stackrel{?}{=} r$$

Frog-cart

Let $L(s) = C(s)H(s) = \frac{N(s)}{D(s)}$.



$$\frac{CH}{1+CH} = \frac{L}{1+L} = \frac{N/D}{1+N/D} = \frac{N}{N+D}$$

$$a \lim_{s\to 0} \frac{CH}{1+CH} = a \lim_{s\to 0} \frac{N(s)}{N(s)+D(s)} = a \frac{N(0)}{N(0)+D(0)}$$

If $N(0) \neq 0$, this is a iff $D(0) = 0$.
Thus, $\lim_{t\to\infty} y = r$ if s is a pole of L.

Summary of fundamentals

System: $a_n y^{(n)}(t) + \cdots + a_0 y(t) = b_m u^{(m)}(t) + \cdots + b_0 u(t)$

Control law: $u(t) =$ scaled $\int$ and $\frac{\mathrm{d}}{\mathrm{d}t}$ of error $r(t) - y(t)$.

$$\Big\downarrow \mathcal{L}$$

$$H(s) := \frac{Y(s)}{U(s)} = \frac{b_m s^m + \cdots + b_0}{a_n s^n + \cdots + a_0} \qquad U(s) = C(s) \underbrace{(R(s) - Y(s))}_{E(s)}$$

$$\frac{Y}{R} = \frac{CH}{1 + CH} = \frac{\cdots}{(s - p_1)^{k_1}(s - p_2)^{k_2} \cdots}, \; p_i \in \mathbb{C}$$

Summary of fundamentals

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$$\lim_{t \to \infty} y(t) = r \text{ if } \forall i, \mathrm{Re}(p_i) < 0, \text{ and } \exists \text{ a pole of } CH \text{ at } s = 0.$$

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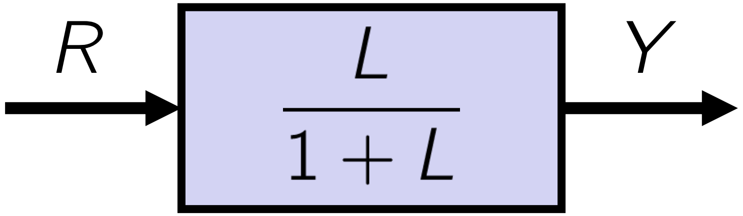
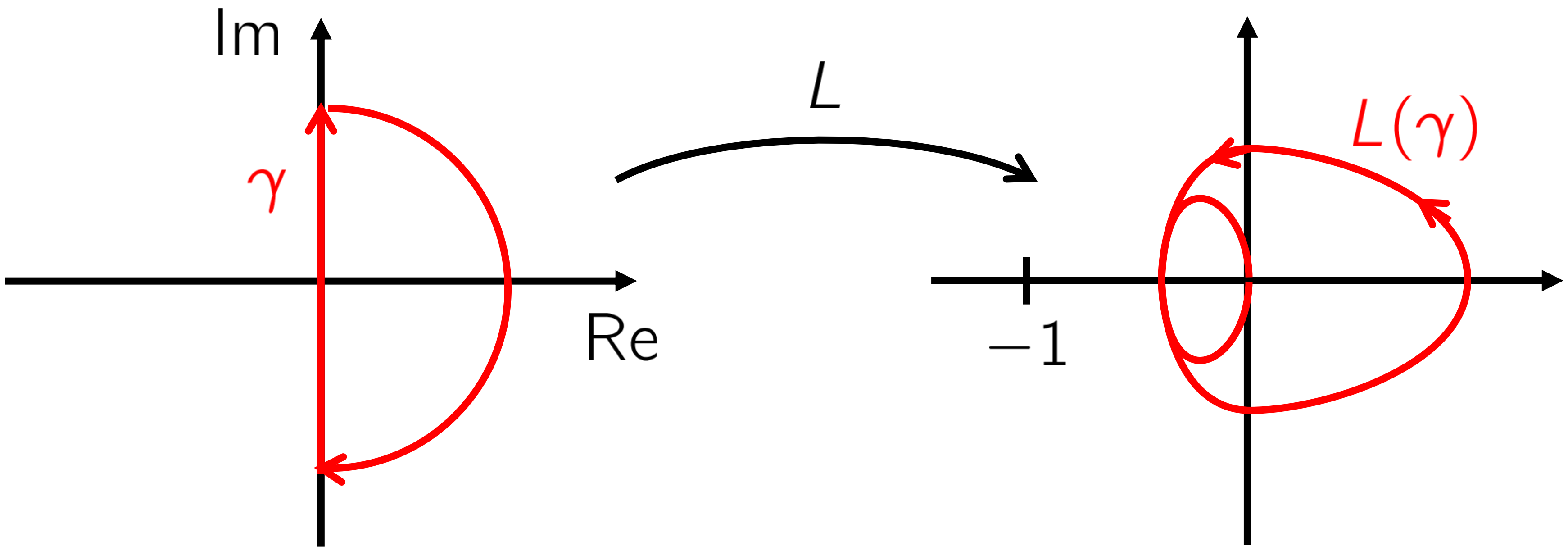
$$\lim_{t\to\infty} y(t) = r \text{ if } \forall i, \mathrm{Re}(p_i) < 0, \text{ and } \exists \text{ a pole of } CH \text{ at } s = 0.$$

Requires us to compute closed-loop transfer function.

A better stability test



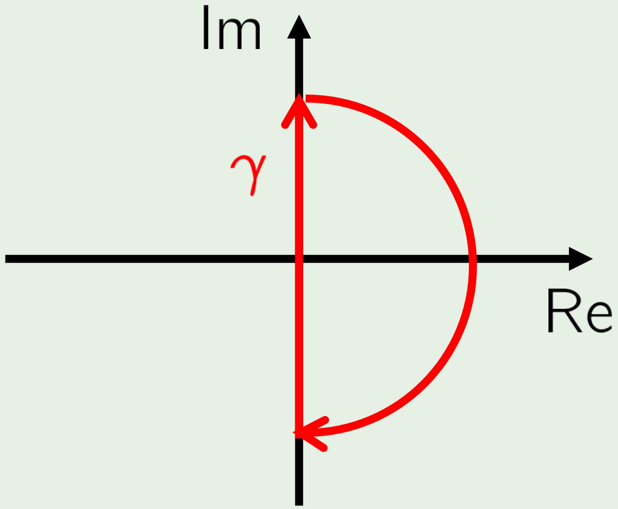
Want to find whether any zeros of $1 + L$ are in the right half-plane $\mathbb{C}_{>}$.



A better stability test

Theorem (Nyquist Stability Criterion)

Define a *Nyquist contour* γ. Then,

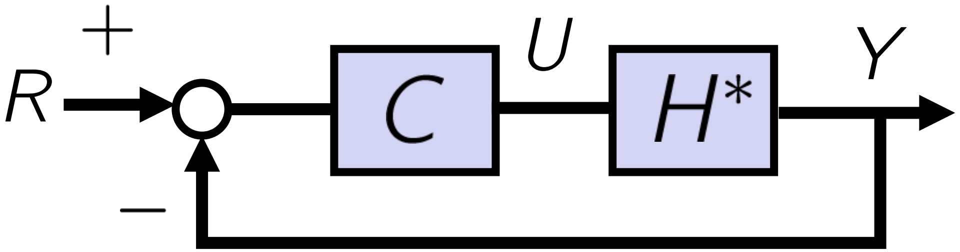


$$\begin{aligned}&\#\text{zeros of } 1 + L \text{ in } \mathbb{C}_> \\ = &\ (\text{winding number of } L(\gamma) \text{ about } -1) \\ &+ (\#\text{poles of } L \text{ in } \mathbb{C}_>)\end{aligned}$$

Only need to evaluate $L(i\omega)$, $\omega \in \mathbb{R}$, to check stability.

Robust control

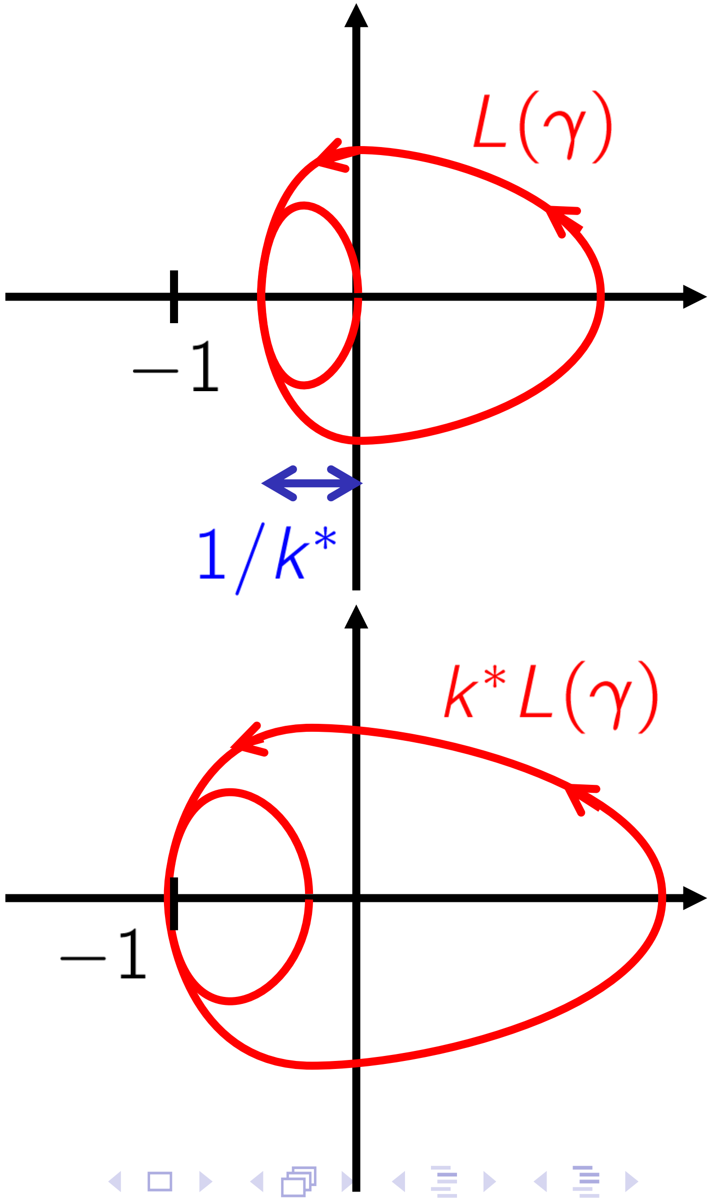
Some systems are naturally unstable.



How much can we "vary" H^* from H while maintaining *stability*?

e.g. $\mathbf{\Delta} = \{kH \mid k \in [0, k^*)\}$

All $H \in \mathbf{\Delta}$ are stabilised by a fixed C.



Robust control

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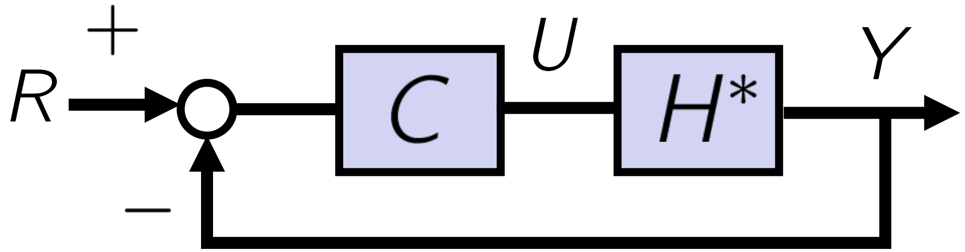

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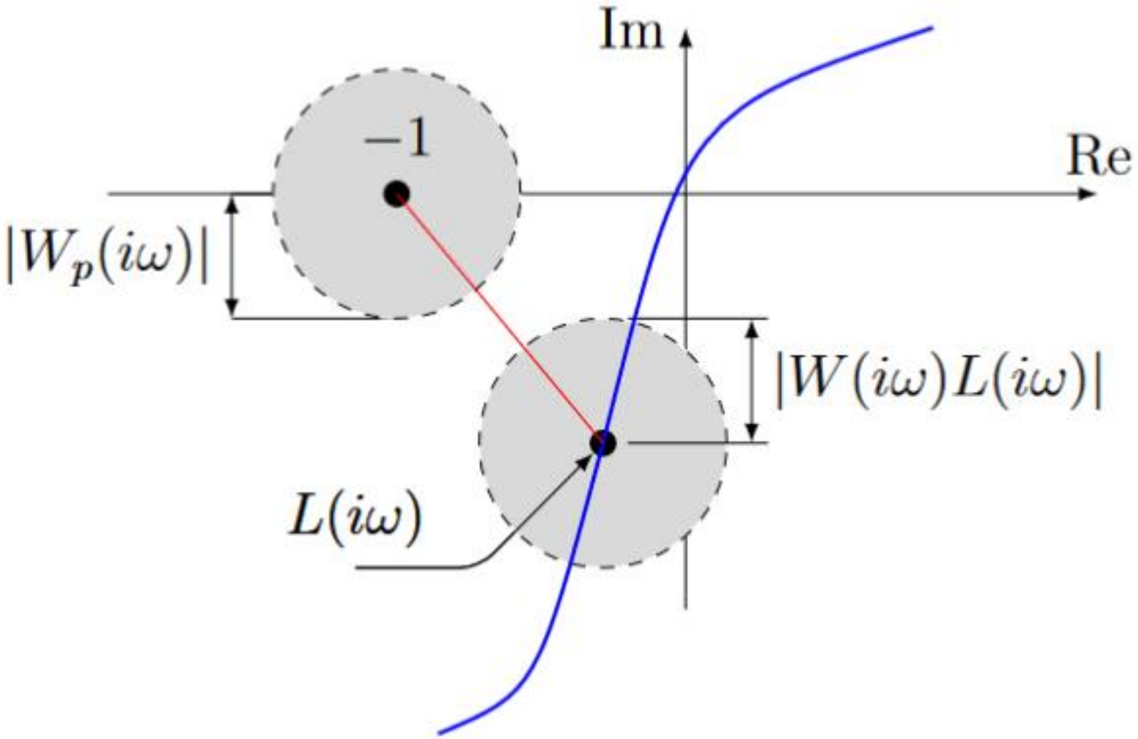


Where to from here?

Optimal control (MATH3404?)

Arbitrary convergence rate!

$$\theta_2(t) = \theta_r(1 - e^{-H^* k_i t})$$

In general, if we specify *performance* requirements, can we meet them?

Reference tracking, disturbance rejection.

Find $\mathbf{K}$ such that $\mathbf{u} = -\mathbf{K}\mathbf{x}$ minimises objective fn.

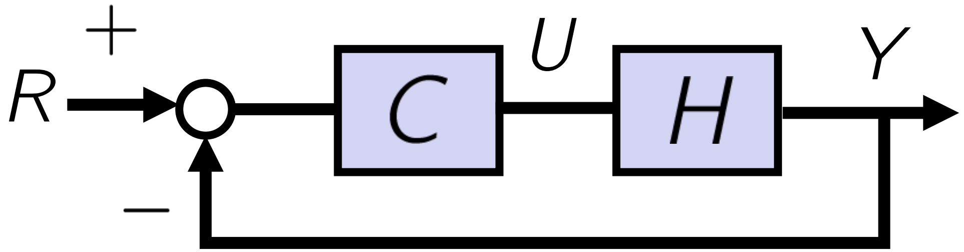
$$J = \int_0^\infty \mathbf{x}^\top \mathbf{Q} \mathbf{x} + \mathbf{u}^\top \mathbf{R} \mathbf{u} \, \mathrm{d}t$$

Thank you for listening! 💖

Nonlinear control

Realisation theory

Geometric control
(e.g. over SO(3))



State estimation

Reinforcement learning

Motion planning

References

Scherer, Carsten. Theory of Robust Control (2018). https://www.imng.uni-stuttgart.de/mst/files/RC.pdf
Aström and Murray. Feedback Systems: An Introduction for Scientists and Engineers (2008).

metr4201 course notes I guess