Simplicial Sets, Simply

Joel Richardson

August 2024

Motivation

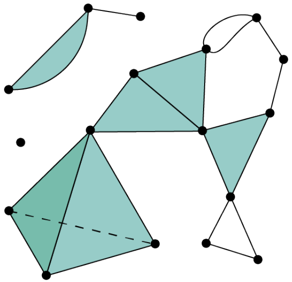

Figure: a cool image

Disclaimer

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...sorry everyone.

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- For each $a \in \mathcal{C}$ there is an element $\mathbf{1}_a \in \mathbf{Hom}(a,a)$ that is an identity with respect to $(\circ)$
- The function $(\circ)$ is associative; $h \circ (g \circ f) = (h \circ g) \circ f$

Categories

Notation

Suppose f is arrow from a to b in some category.
It is cumbersome to write $\mathbf{Hom}^{-1} f = (a,b)$
It is cumbersome to write $f \in \mathbf{Hom}(a,b)$
Instead, write

$$f : a \to b$$

Categories

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- Objects are the sets $[n] = \{0, 1, 2, \ldots, n\}$, for each n.
- Arrows $f : [n] \to [m]$ are the strictly order preserving functions, i.e. f such that $f(i) < f(j)$ if $i < j$.

Categories

Objects	Homomorphisms (Arrows)
sets	functions
vector spaces	linear maps
groups	group homomorphisms
rings	ring homomorphisms
measure spaces	measurable functions
topological spaces	continuous functions

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Example category: for each category $\mathcal{C}$ there is a category $\mathcal{C}^{\mathrm{op}}$

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- Objects are exactly the objects of $\mathcal{C}$
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A **functor** F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is *basically a graph homomorphism*. It consists of a map

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$$\text{if} \quad f\colon x \to y \quad \text{then} \quad Ff : Fx \to Fy$$

Categories - putting it all together

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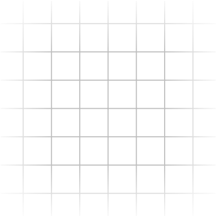
Okay, but like, what?

Breathe. Actual talk begins here.

Definition (geometric simplex)



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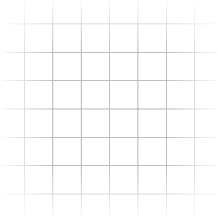
Choose n linearly independent points from $\mathbb{R}^m$.



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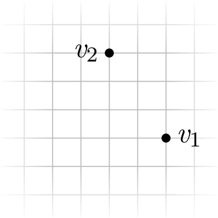

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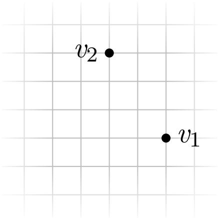

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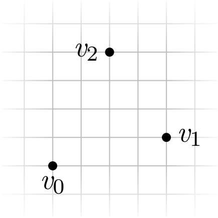

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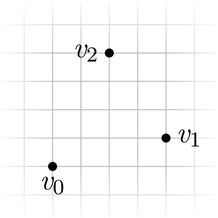

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Such a set is called a **geometric n-simplex**

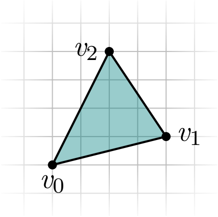

Figure: Geometric 2-simplex

Geometric simplices

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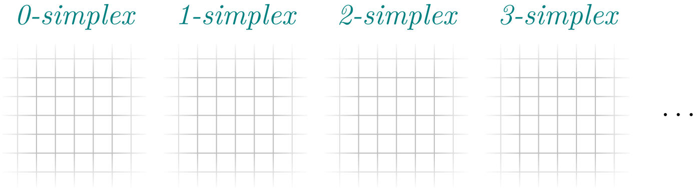

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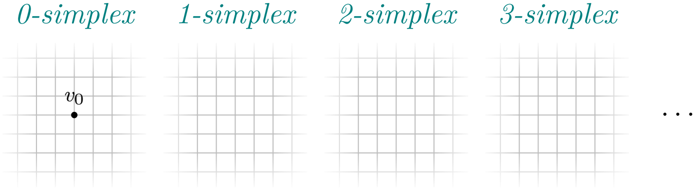

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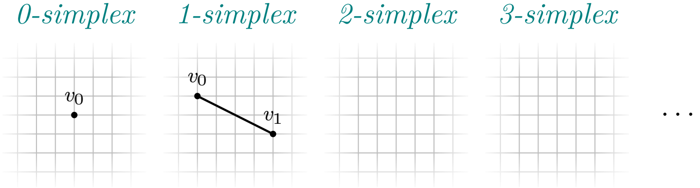

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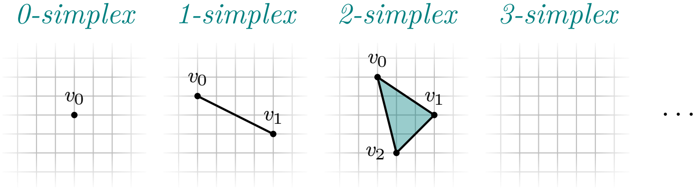

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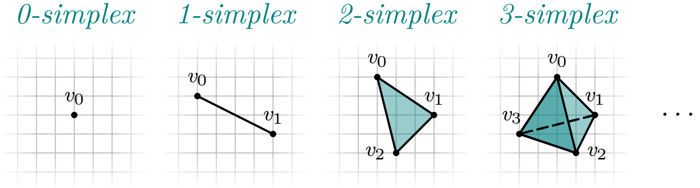

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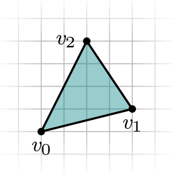

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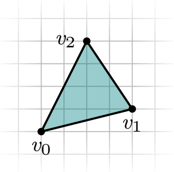


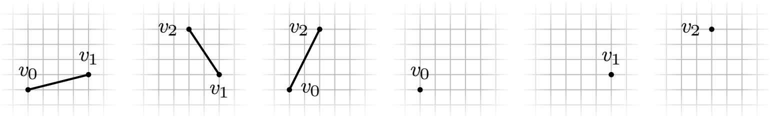


Figure: The faces of a geometric 2-simplex

Definition (simplicial complex)

A simplicial complex is a set X of geometric simplices such that

1. for each $x \in X$ every face of x is in X,
2. for every $x, y \in X$ the intersection $x \cap y$, if non-empty, is a face of each.

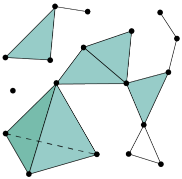


Figure: A simplicial complex

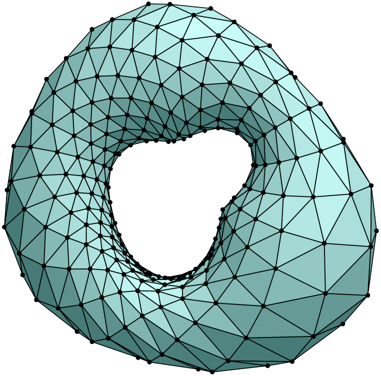
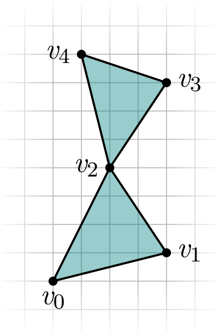


Figure: A torus as a simplicial complex

Moving away from geometry



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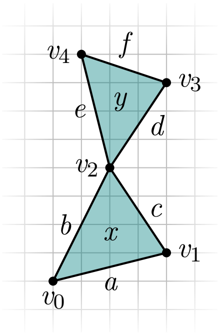

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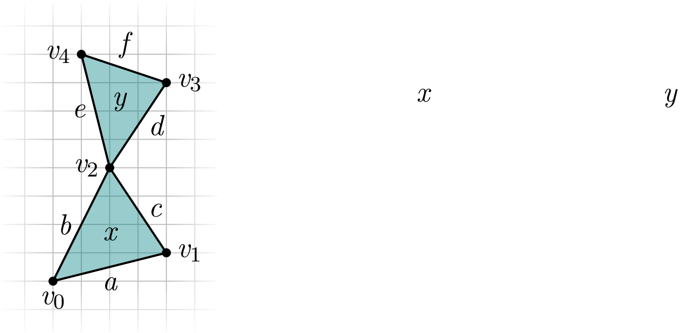

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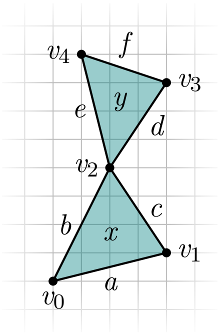

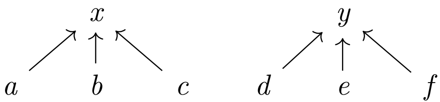

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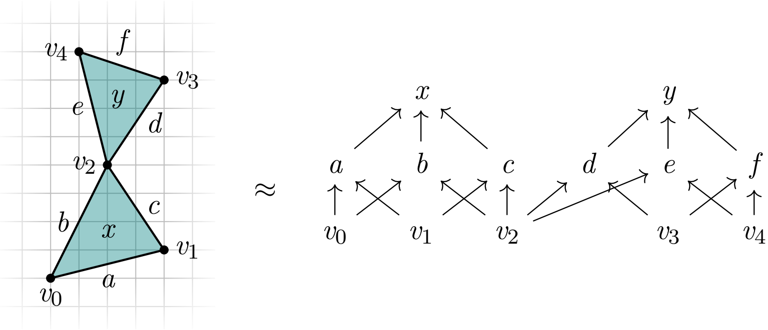

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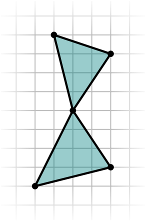

$$\approx \left(\{x, y, a, b, c, d, e, f, v_0, v_1, v_2, v_3, v_4\}, \subseteq \right)$$

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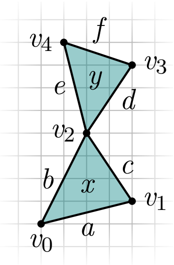
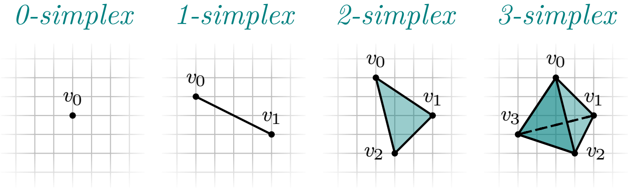
Replace our set X with $(X_n)_{n=1}^{\infty}$



$$X_0 = \{ v_0, v_1, v_2, v_3 \}$$
$$X_1 = \{ a, b, c, d, e, f \}$$
$$X_2 = \{ x, y \}$$
$$X_3 = \varnothing$$
$$\vdots$$

Moving away from geometry

An n-simplex should have $n + 1$ $(n - 1)$-faces; one for each vertex.



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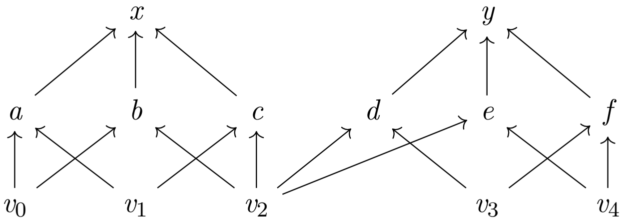
The idea – we replace the subset-arrows with function-arrows. An arrow $x \to a$ says "move from x to a by using deleting vertex blank".



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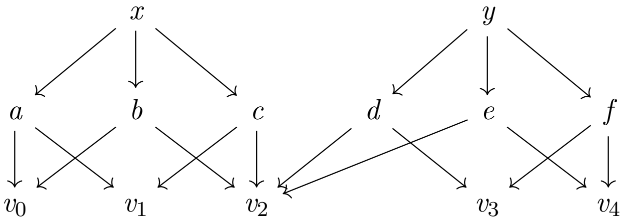

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$$
\begin{array}{c}
\{\, x, y \,\} \\
\downarrow\;\downarrow\;\downarrow \\
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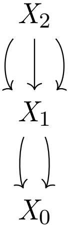

Moving away from geometry

Q: How do we turn two distinct arrows $x \to a$ and $y \to d$ into one arrow $X_2 \to X_1$?

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A: Totally order the set of vertices, X_0.

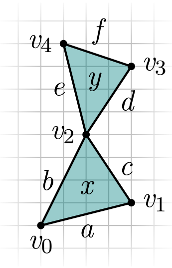
Now we can define a function d_i that *deletes the* i^{th} *smallest vertex*. This can be applied to both x and y.

Moving away from geometry

Totally order X_0 by $v_0 < v_1 < v_2 < v_3 < v_4$. Define functions:

$$d_0^2, d_1^2, d_2^2 : X_2 \to X_1$$
$$d_0^1, d_1^1 : X_1 \to X_0$$

We usually leave off the dimension specification. i.e. $d_i^k(x)$ becomes $d_i(x)$.



$$d_0(x) = c \quad d_0(y) = f$$
$$d_1(x) = b \quad d_1(y) = e$$
$$d_0(a) = v_1 \quad d_0(b) = v_2$$
$$d_1(e) = v_2 \quad d_1(f) = v_3$$
$$(d_1 \circ d_2)(x) = v_0$$

Moving away from geometry

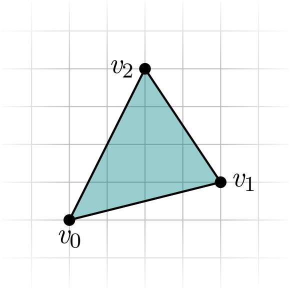


Figure: Geometric 2-simplex

Simplicial sets

A **simplicial set**[1] is a collection of sets $(X_n)_{n=0}^{\infty}$ along with a collection of functions $(d_0^n, \ldots, d_n^n : X_n \to X_{n-1})_{n=1}^{\infty}$ such that

$$\text{for each} \quad i, j \quad \text{if} \quad i < j \quad \text{then} \quad d_i \circ d_j = d_{j-1} \circ d_i$$

[1] This is technically actually a delta-set, but don't sweat it.

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Turns out this is exactly a functor $X : \Delta^{\text{op}} \to \mathbf{Set}$!

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Simplicial sets - What?

A functor $F\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ is

- a set $F[n]$ for each object $[n]$ in Δ
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- a function $Ff\colon F[n] \to F[m]$ for each $f\colon [n] \leftarrow [m]$ in Δ
- *This one is a more involved*
- (such that rules blah blah blah)

Simplicial sets

Consider the functions $D_i^n : [n-1] \to [n]$ with $D_i^n(j) = j$ for $j < i$ and $D_i^n(j) = j + 1$ for $j \geqslant i$.

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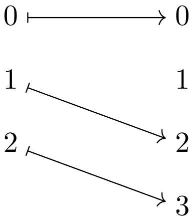

Figure: Pictorial D_1^2

Simplicial sets

Theorem

Every map $f\colon [n] \to [m]$ in Δ is the composition D_i^k functions.

Moreover, $D_j^{n+1} \circ D_i^n = D_i^{n+1} \circ D_{j-1}^n$ if $i < j$

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Proof

Exercise. $\square$

Simplicial sets

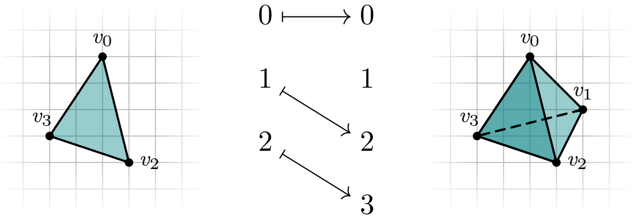

Figure: d_1^3 and D_1^3

Simplicial sets

A functor $F\colon \Delta^{\text{op}} \to \mathbf{Set}$ is

- a set $F[n]$ for each object $[n]$ in Δ
- *Set $F[n] = X_n$ and/or set $X_n = F[n]$*
- a function $Ff\colon F[n] \to F[m]$ for each $f\colon [n] \leftarrow [m]$ in Δ

- (such that rules blah blah blah)

Simplicial sets

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Simplicial sets

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- a function $Ff\colon F[n] \to F[m]$ for each $f\colon [n] \leftarrow [m]$ in Δ
- *Set* $FD_i^n = d_i^n$ *and/or* $d_i^n = FD_i^n$
- (such that rules blah blah blah)
- *(exercise.)*

the end.

the end.

(also every category is a simplicial set)

Simplicial sets

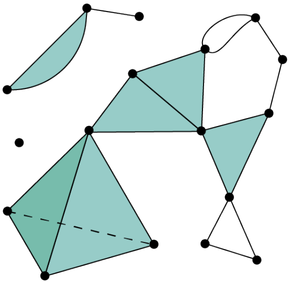

Figure: A simplicial set