The Universal Presence of Category Theory

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Initial thanks

My thanks to:

~~Math~~ *CT* space	for realising the title of this presentation
Peter Gow	for helping me understand CT
MSS	for providing me a platform
You	for listening to me

Why this presentation?

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Main goal

To informally highlight some of the

context behind **basics of** **applications of**

category theory, so that it isn't as intimidating next time.

Outline

Outline

How do graphs fit together?

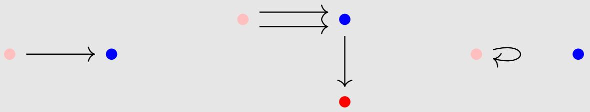
Definition (Graph)

A **graph** G consists of:

a set $V(G)$ of *vertices*, a set $E(G)$ of *edges*

such that each edge has a specified *start* and *end* vertex.

Example



How do graphs fit together?

Remark. We care about the *structure* within a graph, not:

- how the vertices/edges are labelled
- how the vertices/edges are arranged in space

Example

$\bullet \overset{a}{\underset{b}{\rightrightarrows}} \bullet$ is interchangeable with $\bullet \overset{\text{cat}}{\underset{\text{dog}}{\rightrightarrows}} \bullet$

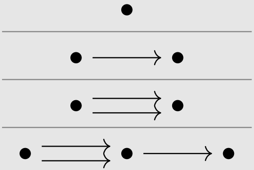
How do graphs fit together?

Question

How do graphs "fit together" amongst each other?

Example

Each graph "forms part of" the graph(s) below it:



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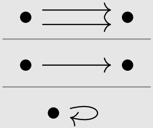
How do graphs "fit together" amongst each other?

Example

Each graph "collapses onto" the graph(s) below it:



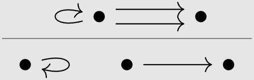
How do graphs fit together?

Question

How do graphs "fit together" amongst each other?

Example

The top graph "collapses onto part of" the graph below it:



How are these relationships encoded?

How do graphs fit together?

Definition (Graph homomorphism)

A **graph homomorphism** $f : G \to H$ consists of:

- A function $V(G) \to V(H)$ (called f)
- A function $E(G) \to E(H)$ (also called f)

such that

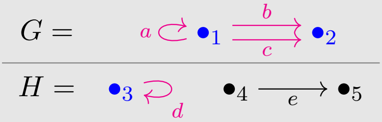
$$
\begin{array}{ccc}
G & \xrightarrow{\ f\ } & H \\
\begin{array}{c} u \\ e \big\downarrow \\ v \end{array} & \xmapsto{\ f\ } & \begin{array}{c} f(u) \\ \big\downarrow f(e) \\ f(v) \end{array}
\end{array}
$$

A graph homomorphism $G \to H$ tells us *how* G "collapses onto part of" H.

How do graphs fit together?

Example

G collapses onto part of H…



…witnessed by $f : G \to H$ with

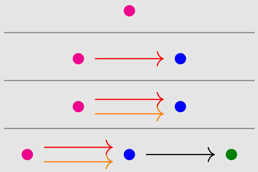
$$f(\bullet_1) = f(\bullet_2) = \bullet_3$$

$$f\left(a \circlearrowleft\right) = f\left(\xrightarrow{b}\right) = f\left(\xrightarrow[c]{}\right) = \circlearrowright d$$

How do graphs fit together?

When $f : G \rightarrowtail H$ is *injective*, there is no “collapse”.
f is a way in which “H is seen from the perspective of G”.

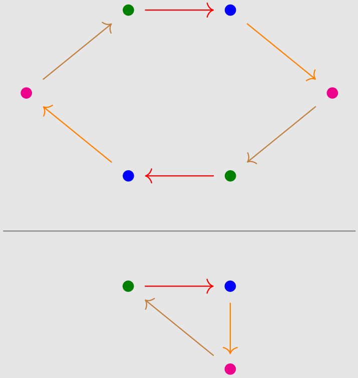
Example



How do graphs fit together?

When $f : G \twoheadrightarrow H$ is *surjective*, then “H is a (perhaps degenerate) version of G”.

Example



A 3-cycle, traversed twice, is a degenerate 6-cycle.

How do graphs fit together?

Observation

Graph homomorphisms give us a powerful way to understand how graphs mimic one another.

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Graph homomorphisms give us a powerful way to understand how graphs mimic one another.

The same game can be played with:

- Groups and group homomorphisms (e.g. $\mathbb{Z}_{/4} \twoheadrightarrow \mathbb{Z}_{/2}$)
- Topological spaces and continuous maps (e.g. paths are maps out of $[0, 1]$)
- Sets and functions (e.g. $|X| \leq |Y|$ iff $X \rightarrowtail Y$)

In all cases, maps are more important than objects.

Definition (Category)

A **category** $\mathcal{C}$ consists of:

- A collection $\mathrm{ob}(\mathcal{C})$ of **objects**
- $\forall a, b \in \mathrm{ob}(\mathcal{C})$, a collection $\mathcal{C}(a, b)$ of **arrows** from a to b
- $\forall a \in \mathrm{ob}(\mathcal{C})$, an **identity arrow** $1_a \in \mathcal{C}(a, a)$
- $\forall a, b, c \in \mathrm{ob}(\mathcal{C})$, a **composition function**

$$\mathcal{C}(b, c) \times \mathcal{C}(a, b) \xrightarrow{\circ} \mathcal{C}(a, c)$$
$$(g, f) \longmapsto g \circ f$$

Such that for all $a \xrightarrow{f} b \xrightarrow{g} c \xrightarrow{h} d$ in $\mathcal{C}$,

$$1_b \circ f = f \circ 1_a = f \qquad h \circ (g \circ f) = (h \circ g) \circ f$$

Remark. The definition of a category emphasises the arrows. "Whenever you introduce new objects, you should specify the arrows between them."

Example

$\mathbf{Set}$ has:

- objects are *sets*
- arrows $X \to Y$ are *functions* $X \to Y$
- identity arrows id_X are *identity functions*
- composition $g \circ f$ is *function composition*

"Structures and structure-preserving maps":

$\mathbf{Monoid}$	$\mathbf{Grp}$	$\mathbf{Ab}$
$\mathbf{Vect}$	$\mathbf{Graph}$	$\mathbf{Top}$

Categories

Example

A poset $(P, \leq)$ determines a category $\mathcal{P}$ where:

- objects are *elements of* P
- if $x \leq y$, then there is a unique arrow $x \to y$ in $\mathcal{P}$
- identity arrows correspond to reflexivity $(x \leq x)$
- composites correspond to transitivity $(x \leq y \leq z \implies x \leq z)$

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A group $(G, \cdot)$ determines a category $\mathrm{B}G$ where:

- there is one object $*$
- $\mathrm{B}G(*, *) = G$
- the identity 1_* is the identity element $e \in G$
- composition $h \circ g := h \cdot g$ uses the group operation

Categories

Category theory provides language and insight for compositionality.

Instantiating general category-theoretic ideas in any particular category often yields interesting results.

Outline

Monomorphisms and epimorphisms

Recall that an *injective* graph hom. captures *substructure*, whereas a *surjective* graph hom. captures *(degenerate) quotient structure*.

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$$\forall \left(x \underset{g}{\overset{f}{\rightrightarrows}} a \xrightarrow{m} b \right), \quad m \circ f = m \circ g \implies f = g$$

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Dually, an arrow $a \xrightarrow{e} b$ is an **epimorphism** just when

$$\forall \left(a \xrightarrow{e} b \overset{f}{\underset{g}{\rightrightarrows}} x \right), \quad f \circ e = g \circ e \implies f = g$$

Monomorphisms and epimorphisms

Example

In $\mathbf{Set}$, any subset yields an inclusion map:

$$A \subseteq X \implies \begin{array}{rcl} A & \xrightarrow{\iota} & X \\ a & \longmapsto & a \end{array}$$

This map is *monic*.

Monomorphisms and epimorphisms

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In $\mathbf{Grp}$, any normal subgroup yields a quotient map:

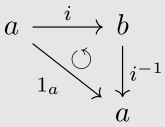
$$N \trianglelefteq G \implies \begin{aligned} G &\xrightarrow{\pi} G/N \\ g &\longmapsto gN \end{aligned}$$

This map is *epic*.

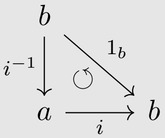
Isomorphisms

Definition (Isomorphism)

An arrow $a \xrightarrow{i} b$ is an **isomorphism** just when $\exists b \xrightarrow{i^{-1}} a$ such that



and



(i.e. $i^{-1} \circ i = 1_a$ and $i \circ i^{-1} = 1_b$.)

Example

Set	Bijection
Top	Homeomorphism
Grp	Isomorphism
Graph	Isomorphism

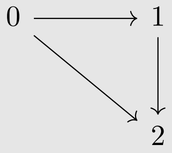
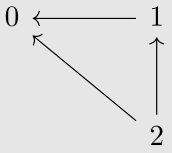
Duality

Definition (Opposite category)

Given a category $\mathcal{C}$, its **opposite category** $\mathcal{C}^{\text{op}}$ consists of:

- Objects: same as $\mathcal{C}$
- Arrows $a \to b$ in $\mathcal{C}^{\text{op}}$: arrows $b \to a$ in $\mathcal{C}$
- Identities: same as $\mathcal{C}$
- Composition: $g \circ^{\text{op}} f := f \circ g$.

Example

$\mathcal{C} = (\{0,1,2\}, \leq)$	$\mathcal{C}^{\text{op}} = (\{0,1,2\}, \geq)$
	

Duality

"Interesting" structures in $\mathcal{C}^{\mathrm{op}}$ are "interesting" in $\mathcal{C}$, too.

Example

Let $b \xrightarrow{e} a$ be a monomorphism in $\mathcal{C}^{\mathrm{op}}$. That is,

$$\forall \left(x \underset{g}{\overset{f}{\rightrightarrows}} b \xrightarrow{e} a \right) \text{ in } \mathcal{C}^{\mathrm{op}}, \quad e \circ^{\mathrm{op}} f = e \circ^{\mathrm{op}} g \implies f = g$$

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Turning all the arrows around,

$$\forall \left(a \xrightarrow{e} b \underset{g}{\overset{f}{\rightrightarrows}} x \right) \text{ in } \mathcal{C}, \quad f \circ e = g \circ e \implies f = g$$

So a monomorphism in $\mathcal{C}^{\mathrm{op}}$ is an epimorphism in $\mathcal{C}$.

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So a monomorphism in $\mathcal{C}^{\mathrm{op}}$ is an epimorphism in $\mathcal{C}$.

Remark. An iso. in $\mathcal{C}^{\mathrm{op}}$ is an iso. in $\mathcal{C}$.

Duality

Duality principle

In any statement that says "$\forall$ categories $\mathcal{C}$", you can replace $\mathcal{C}$ with $\mathcal{C}^{\mathrm{op}}$.

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Example

Exercise: in any category $\mathcal{C}$, any isomorphism is monic.

$\Downarrow$ dual (isomorphism) $\Downarrow$ dual (monic)

Therefore: in any category $\mathcal{C}$, any isomorphism is epic.

Functorality

Functorality and naturality were the motive for the first paper in category theory [1].

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Definition (Functor)

Let $\mathcal{C}, \mathcal{D}$ be categories. A **functor** $F : \mathcal{C} \to \mathcal{D}$ consists of:

- A function $\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D})$ (called F)
- $\forall c, c' \in \mathcal{C}$, a function $\mathcal{C}(c, c') \to \mathcal{D}(F(c), F(c'))$ (also F)

subject to:

$$F(1_c) = 1_{F(c)} \qquad\qquad F(g \circ f) = F(g) \circ F(f)$$

Functorality

Example

$$\begin{array}{ccc} \mathbf{Graph} & \xrightarrow{V} & \mathbf{Set} \\ G & & V(G) \\ f \downarrow & \longmapsto & \downarrow f \\ H & & V(H) \end{array}$$

Observe:

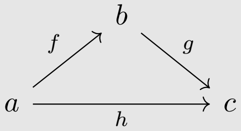
$$1_G \overset{V}{\longmapsto} 1_{V(G)}$$
$$g \circ f \overset{V}{\longmapsto} g \circ f$$

so retrieving vertex sets is *functorial*.

Functorality

Definition (Commutative diagram)

A **commutative diagram** in $\mathcal{C}$ is a graph of objects and arrows, e.g.



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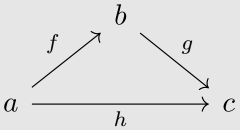

such that any two parallel paths have equal composites (e.g. $g \circ f = h$).

Lemma (Functors preserve commutative diagrams)

Take a commutative diagram in $\mathcal{C}$, and apply a functor $F : \mathcal{C} \to \mathcal{D}$ to all objects and arrows in the diagram. Then, the resulting diagram commutes in $\mathcal{D}$.

Functorality

Lemma (Functors preserve commutative diagrams)

Example

An isomorphism $i : a \simeq b$ in $\mathcal{C}$:

$$\begin{array}{ccc} a & \xrightarrow{i} & b \\ & {\scriptstyle 1_a} \searrow & \downarrow {\scriptstyle i^{-1}} \\ & & a \end{array} \quad \text{and} \quad \begin{array}{ccc} b & & \\ {\scriptstyle i^{-1}} \downarrow & \searrow {\scriptstyle 1_b} & \\ a & \xrightarrow[i]{} & b \end{array} \quad \text{in } \mathcal{C}$$

Applying a functor $F : \mathcal{C} \to \mathcal{D}$, we get *commutative* diagrams

$$\begin{array}{ccc} F(a) & \xrightarrow{F(i)} & F(b) \\ & {\scriptstyle 1_{F(a)}} \searrow & \downarrow {\scriptstyle F(i^{-1})} \\ & & F(a) \end{array} \quad \text{and} \quad \begin{array}{ccc} F(b) & & \\ {\scriptstyle F(i^{-1})} \downarrow & \searrow {\scriptstyle 1_{F(b)}} & \\ F(a) & \xrightarrow[F(i)]{} & F(b) \end{array} \quad \text{in } \mathcal{D}$$

which says that $F(i)$ is an isomorphism $F(a) \simeq F(b)$ in $\mathcal{D}$.

Functorality

Any category $\mathcal{C}$ has an *identity functor*

$$1_{\mathcal{C}} : \quad c \xrightarrow{f} d \quad \longmapsto \quad c \xrightarrow{f} d$$

Functors $\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ compose:

$$\left(a \xrightarrow{f} a'\right) \quad \xmapsto{G \circ F} \quad \left(G(F(a)) \xrightarrow{G(F(f))} G(F(b))\right)$$

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Definition (Cat)

Cat is the category with

- objects: categories
- arrows: functors

and the above identities and composites.

Naturality

"Whenever you introduce new objects, you should specify the arrows between them."

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Definition (Natural transformation)

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A **natural transformation** $\alpha : F \Rightarrow G$ consists of:

- A family $\alpha_c : F(c) \to G(c)$ of arrows in $\mathcal{D}$, for each $c \in \mathcal{C}$

such that

$$\forall a \xrightarrow{\forall f} \forall b \qquad \text{in } \mathcal{C}$$

$$\begin{array}{ccc} F(a) & \xrightarrow{F(f)} & F(b) \\ {\scriptstyle \alpha_a}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_b} \\ G(a) & \xrightarrow[G(f)]{} & G(b) \end{array} \qquad \text{in } \mathcal{D}$$

commutes.

Naturality

Remark. **Cat** has:

- objects $\mathcal{C}, \mathcal{D}, \ldots$
- arrows $F, G, \ldots : \mathcal{C} \to \mathcal{D}$
- *2-cells* $\alpha, \ldots : F \Rightarrow G$

which makes it a 2-category.

Universality

$$\forall \text{thing}, \exists! \text{otherThing} : \text{ condition}$$

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Example

Let $V, W \in \mathbf{Vect}_K$ and let β be a basis for V.

$$\forall f : \beta \xrightarrow{\text{function}} W,$$

$$\exists! T : V \xrightarrow{\text{linear}} W :$$

$$T|_\beta = f$$

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$$\forall f : \beta \xrightarrow{\text{function}} W,$$
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$$T|_\beta = f$$

$$\mathbf{Vect}_K\left(\operatorname{span}_K(\beta), W\right) \simeq \mathbf{Set}(\beta, W)$$

natural in β, W; **adjunction**.

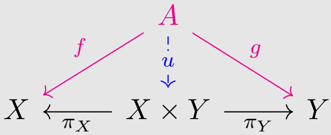
Universality

$$\forall \text{thing}, \exists! \text{otherThing} : \text{ condition}$$

Example

Let $X, Y \in \mathbf{Set}$. The product projections $X \xleftarrow{\pi_X} X \times Y \xrightarrow{\pi_Y} Y$ satisfy

$\forall A, \forall f, \forall g, \exists! u :$



commutes

$(u : a \mapsto (f(a), g(a)).)$ **Limit**.

Remark. The p-adic numbers are the colimit of $\mathbb{Z}_{/p} \hookrightarrow \mathbb{Z}_{/p^2} \hookrightarrow \cdots$

$$\forall \text{thing}, \exists! \text{otherThing} : \text{condition}$$

Example

There is a bijection

$$(n\text{-colour vertex colourings})(G) \simeq \mathbf{Graph}(G, K_n)$$

natural in G. Vertex colourings are **representable**, and form the **universal property** of K_n.

All the universal properties come in this last form.

Yoneda lemma

Scary version:

$$\mathbf{Set}^{\mathcal{C}}\left(\mathcal{C}(x,-), F\right) \simeq_{\mathsf{nat}.F,x} F(x)$$

Useful version:

Lemma (Yoneda)

$$\begin{aligned} &\mathcal{C}(a,x) \simeq_{\mathsf{nat}.x} \mathcal{C}(b,x) \\ \implies \quad & a \simeq_{\mathcal{C}} b \end{aligned}$$

Consequences:

- "Equational" proofs
- Easier handling of universal properties

Outline

Functional programming

Functional programming is a paradigm where nothing can change value.

Example

Imperative program:

```
x := 1; printf("%d", x);
x := 2; printf("%d", x);
// Wdym; 1 != 2?!
// This is clearly evil.
```

Functional programming

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Functional program:

```
main = do
  putStrLn "1"
  putStrLn "2"
-- Much "better"
```

Functional programming

Example

Loops aren't functional:

```
for(int i = 0; i < 10; i++) {
  printf("%d", i);
}
// Variable i changes value!
```

Recursion is functional:

```
printer 9 = putStrLn "9"
printer n = putStrLn (show n) >> printer (n + 1)
main = printer 0
```

Functional programming languages are *very good* for recursion.

Functional programming

Imperative:	Functional:
Data types	Data types
Functions	Functions
Variables	*More* functions
Loops	**More** functions
Error handling	**More** functions
. . .	. . .

Functional programming

Imperative:	Functional:
Data types	Data types
Functions	Functions
Variables	*More* functions
Loops	**More** functions
Error handling	**More** functions
...	...

Reasoning about imperative programs is hard.
Reasoning about functional programs is easier.

Categorical influence

Haskell has an associated category $\mathbf{Hask}$, with

- Objects: Data types $A, B, \ldots$
- Arrows $A \to B$: *Computable* functions $A \to B$

Pretend $\mathbf{Hask} = \mathbf{Set}$ if you want to.

For technical (cringe) reasons, this isn't actually a category.

Categorical influence: Algebraic data types

There are some built-in data types:

- `Integer`
- `Char`
- etc.

There's a really cool way to build more.

Example

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- etc.

There's a really cool way to build more.

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data IntegerAndChar = ThePair Integer Char
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Categorical influence: Algebraic data types

There are some built-in data types:

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Polymorphic data types are functors:

$$\texttt{Pair} : \mathbf{Hask} \times \mathbf{Hask} \longrightarrow \mathbf{Hask}$$
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Example

```
(+1) <$>     [] =     []
(+1) <$>   1:[] =   2:[]
(+1) <$> 1:2:[] = 2:3:[]
```

Categorical influence: Algebraic data types

Example

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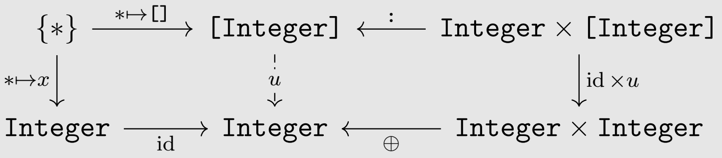
Polymorphic data types have a universal property:

Example

$\forall x, \forall \oplus, \exists! u :$



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Example

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$$
\begin{array}{ccccc}
\{*\} & \xrightarrow{* \mapsto []} & [\texttt{Integer}] & \xleftarrow{:} & \texttt{Integer} \times [\texttt{Integer}] \\
{\scriptstyle * \mapsto x}\downarrow & & \downarrow{\scriptstyle !\, u} & & \downarrow{\scriptstyle \mathrm{id} \times u} \\
\texttt{Integer} & \xrightarrow[\mathrm{id}]{} & \texttt{Integer} & \xleftarrow[\oplus]{} & \texttt{Integer} \times \texttt{Integer}
\end{array}
$$

With $x := 0$ and $\oplus := +$, we get $u = \texttt{sum}$.

Categorical influence: Monads

Error handling in Haskell is sick.

Maybe

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data Maybe a = Nothing | Just a
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As a functor, `Maybe` : $\mathbf{Hask} \to \mathbf{Hask}$.

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data Maybe a = Nothing | Just a
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As a functor, $\texttt{Maybe} : \mathbf{Hask} \to \mathbf{Hask}$.
But really, `Maybe` fakes the behaviour

$$\mathbf{Hask} \longrightarrow \mathbf{Hask}_*$$

by sending A to $(A \sqcup \{\texttt{Nothing}\}, \texttt{Nothing})$.
The categorical way of "faking" this behaviour is a monad.

Categorical influence: Monads

Example

```
main = do
  putStrLn "gimme a number"
  number <- (read @Integer) <$> getLine
  putStrLn $ "you said " ++ show number

>> main
out> gimme a number
in>  69
out> you said 69
```

Monoidal categories

Categories tell us about moving from one object to another.
Monoidal categories tell us about combining objects together.

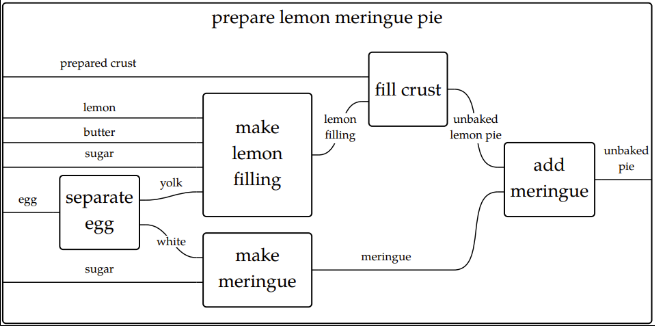

Figure: Wiring diagram for preparing a lemon meringue pie

src: Fong and Spivak's *Seven Sketches* [2]

Enrichment

Monoidal categories serve as bases for enriched categories.

Definition ($\mathcal{V}$-Category)

Fix a monoidal category $\mathcal{V}$.
A $\mathcal{V}$-**category** $\mathcal{C}$ consists of:

- A collection $\mathrm{ob}(\mathcal{C})$ of **objects**
- $\forall a, b \in \mathrm{ob}(\mathcal{C})$, a **hom-object** $\mathcal{C}(a, b) \in \mathcal{V}$
- *(and identities and composition)*

subject to some coherence conditions

Key point: Replace set of arrows $\mathcal{C}(a, b)$ with $\mathcal{V}$-thing of arrows $\mathcal{C}(a, b)$.

Enrichment

Example

$\mathbf{Bool} = (\texttt{false} \to \texttt{true})$ is a monoidal category.
A $\mathbf{Bool}$-category is a preorder:

$$a \leq b \iff \mathcal{C}(a, b) = \texttt{true}$$

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Example

An $\mathbf{Ab}$-category is an **additive category**: $\mathcal{C}(a, b)$ is an abelian group, so arrows $f, g : a \to b$ have a *sum* $f + g : a \to b$.

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Example

An $\mathbf{Ab}$-category is an **additive category**: $\mathcal{C}(a, b)$ is an abelian group, so arrows $f, g : a \to b$ have a *sum* $f + g : a \to b$.

Example

A $\mathbf{Cat}$-category is a 2-category; e.g. $\mathbf{Cat}$.

Abelian categories

Definition (Abelian category)

An **abelian category** is an $\mathbf{Ab}$-category with the **first isomorphism theorem**.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow\!\!\downarrow & & \uparrow \\ \operatorname{coim}(f) & \overset{\simeq}{\dashrightarrow} & \operatorname{im}(f) \end{array}$$

These have use in homological algebra, especially in cohomology [3].

Outline

Conclusions

Main goal

To informally highlight some of the

context behind **basics of** **applications of**

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Context:

- Studying functions
- Regarding arrows as more important than objects

Conclusions

Main goal

To informally highlight some of the

context behind **basics of** **applications of**

category theory, so that it isn't as intimidating next time.

Basics:

- Monos, epis, isos
- Duality
- Functors and natural transformations
- Universality
- Yoneda lemma

Conclusions

Main goal

To informally highlight some of the

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category theory, so that it isn't as intimidating next time.

Applications:

- Functional programming, esp. in the Haskell type system
- Monoidal categories and "combining objects"
- Enrichment and abelian categories, esp. in homology

Terminal thanks

Thanks 4 watching.

References.

[1] Samuel Eilenberg and Saunders Maclane. "General theory of natural equivalences". In: *Transactions of the American Mathematical Society* 58 (1945), pp. 231–294. URL: https://api.semanticscholar.org/CorpusID:10673176.

[2] Brendan Fong and David I. Spivak. *An Invitation to Applied Category Theory: Seven Sketches in Compositionality*. Cambridge University Press, 2019.

[3] Alexander Grothendieck. "Sur quelques points d'algèbre homologique, I". In: *Tohoku Mathematical Journal* 9.2 (1957), pp. 119–221. DOI: 10.2748/tmj/1178244839. URL: https://doi.org/10.2748/tmj/1178244839.