Self-Similarity via Attractors

Gabriel Field

28/April/2023

Outline

1. Goals
2. Motivation (Pretty Pictures)
3. Iterated Function Systems and Self-Similarity
4. The Hausdorff Metric
5. Existence of Self-Similar Shapes
6. Aside: Fractals

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Goals

Main goal: You should leave this room convinced that *a self-similar set exists*.

Other goals:

- Give you a taste of basic fractal geometry
- Give you an excuse to look at pretty pictures
- Give you some familiarity with the tools and setting used

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Motivation (Pretty Pictures)

We'll consider these...

One Billion Pyramids - Sierpinski 3D Fractal Trip

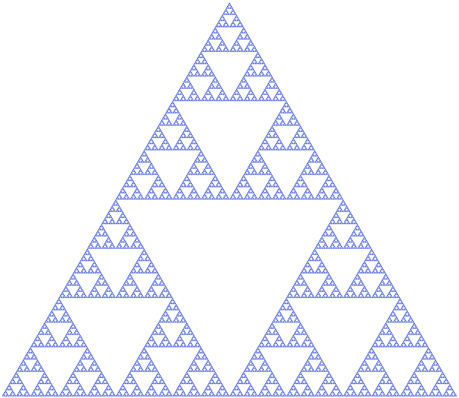

Figure: Sierpinski Triangle

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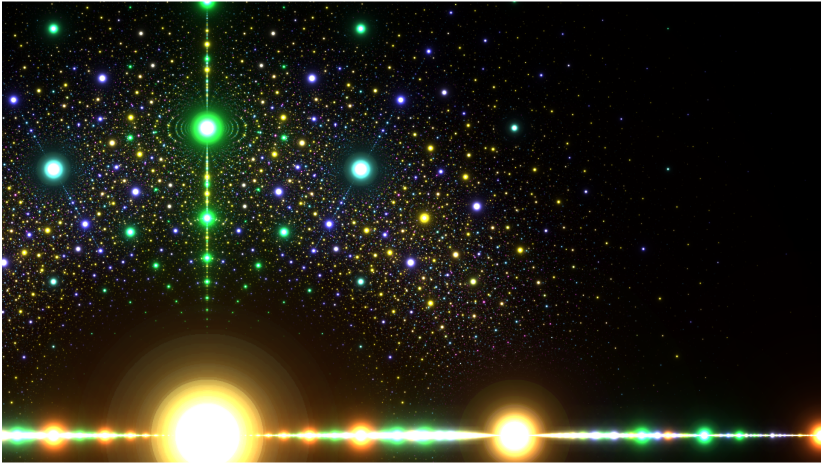

Figure: Algebraic Numbers

Motivation (Pretty Pictures)

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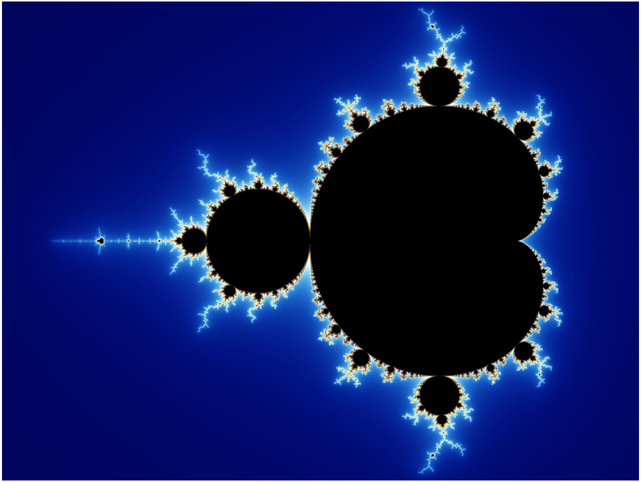

Figure: Mandelbrot Set

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Spaces and Shapes

We work in a *metric space*.

Definition

A *metric space* is a pair (X, d) where $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfying that for all $x, y \in X$,

(this property): $d(x, y) = 0$ iff $x = y$

(symmetry): $d(x, y) = d(y, x)$

(triangle inequality): $d(x, y) \leq d(x, z) + d(z, y)$

Examples

$(\mathbb{R}^n, d)$ with $d : (x, y) \mapsto \|x - y\|$
Most things

A *shape* is any subset of X.

Towards IFSes

We want to construct a self-similar shape.
How do we do it?

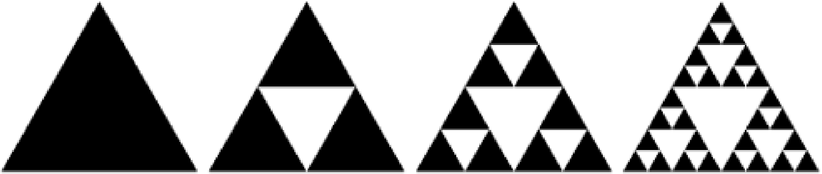

Figure: Construction of Sierpinski triangle

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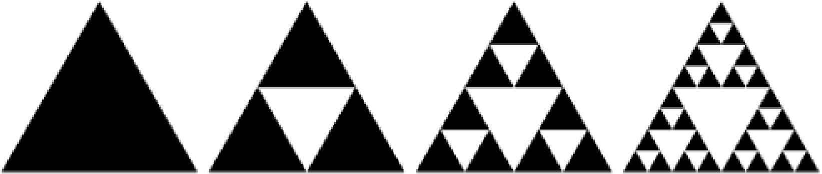

Figure: Construction of Sierpinski triangle

'Cutting out' seems appropriate, but doesn't always work (see next slide)

Towards IFSes

Example where 'cutting out' fails:

Figure: Barnsley Fern

Towards IFSes

Example where 'cutting out' fails:

Figure: Barnsley Fern

Solution: use contraction maps to encode 'self-similar copies'.

Lipschitz Maps

Definition

Given metric spaces (X, d_X), (Y, d_Y) and $L \geq 0$,
A map $\psi : X \to Y$ is called *Lipschitz with constant* L if for all $x, x' \in X$,

$$d_Y(\psi(x), \psi(x')) \leq L\, d_X(x, x')$$

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The *Lipschitz constant* of a Lipschitz map ψ is

$$\mathrm{Lip}(\psi) = \inf\{L \geq 0 \mid \psi \text{ is Lipschitz with constant } L\}$$

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Remark

Any Lipschitz map ψ is Lipschitz with constant $\mathrm{Lip}(\psi)$.

Contraction Maps

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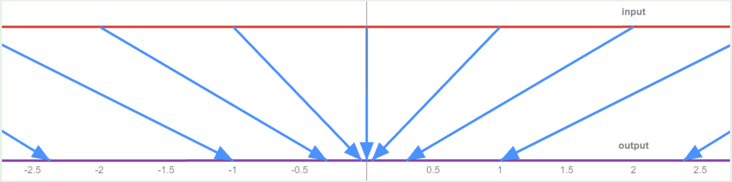
Equivalently, ψ is a contraction if it is Lipschitz and $\mathrm{Lip}(\psi) < 1$.

Examples

$\psi : [-2.5, 2.5] \to \mathbb{R},\ x \mapsto (x/3)^3$ has $\mathrm{Lip}(\psi) \approx 0.69444\cdots$.



Contraction Maps: More Examples

Examples

The maps

$$\psi_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad \psi_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \quad \psi_3 : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$x \longmapsto \frac{1}{2}x \quad x \longmapsto \frac{1}{2}x + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \quad x \longmapsto \frac{1}{2}x + \begin{pmatrix} 1/4 \\ \sqrt{3}/2 \end{pmatrix}$$

are contractions with $\mathrm{Lip}(\psi_i) = 1/2$ (for each i).

See these visualisations of the ψ_i.

Iterated Function Systems

Definition

Given a metric space (X, d),
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Given an IFS Ψ on a metric space (X, d),
The *iteration map* is the function

$$\begin{aligned} \Psi^1 : \mathcal{H}(X) &\longrightarrow \mathcal{H}(X) \\ A &\longmapsto \bigcup_{\psi \in \Psi} (\psi(A)) \end{aligned}$$

(We will define the object $\mathcal{H}(X) \subseteq \mathcal{P}(X)$ later.)

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The idea: use contraction maps to encode the 'self-similar copies' in a self-similar shape.

IFS Example

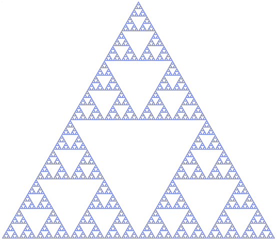

Figure: Sierpinski Triangle $\mathcal{S}$

Origin at lower-left corner of $\mathcal{S}$. Then, $\mathcal{S}$ is described by the IFS $\Psi = \{\psi_1, \psi_2, \psi_3\}$.

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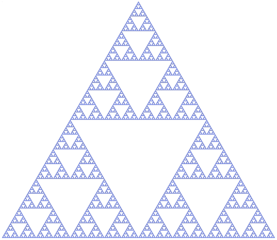

Figure: Sierpinski Triangle $\mathcal{S}$

Origin at lower-left corner of $\mathcal{S}$. Then, $\mathcal{S}$ is described by the IFS $\Psi = \{\psi_1, \psi_2, \psi_3\}$.

$$\begin{aligned} \psi_1 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto \frac{1}{2}x \end{aligned} \qquad \begin{aligned} \psi_2 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto \frac{1}{2}x + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix} \end{aligned} \qquad \begin{aligned} \psi_3 : \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ x &\longmapsto \frac{1}{2}x + \begin{pmatrix} 1/4 \\ \sqrt{3}/2 \end{pmatrix} \end{aligned}$$

Self-Similarity

We can finally state what *self-similarity* is.

Definition

Given a metric space (X, d),
A (non-empty, compact) subset $A \in \mathcal{H}(X)$ is called *self-similar* if there exists an IFS Ψ on X such that $A = \Psi^1(A)$.

So, a self-similar set is one which is fixed by the iteration map of some IFS.

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So, a self-similar set is one which is fixed by the iteration map of some IFS. It's not yet obvious that a self-similar set exists. We need more machinery to prove that.

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When we look at the behaviour of the iteration map Ψ^1, it would be useful to track the 'distance between sets'.

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First, we restrict to a particular class of sets which we can precisely measure the distance between.

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Set $\mathcal{H}(X) = \{A \subseteq X \mid A \text{ is non-empty and compact}\}$.

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Some authors use $\mathcal{H}(X)$ as the collection of subsets which are non-empty, closed and bounded. Conventions are equivalent in Euclidean spaces.

Point-to-set distance $d_{\mathrm{p-S}}$

First measure the distance from a point to a set...

Definition

Given a metric space (X, d), $a \in X$ and $B \in \mathcal{H}(X)$,
The *point-to-set distance* from a to B is $d_{\mathrm{p-S}}(a, B) = \inf_{b \in B}\{d(a, b)\}$.

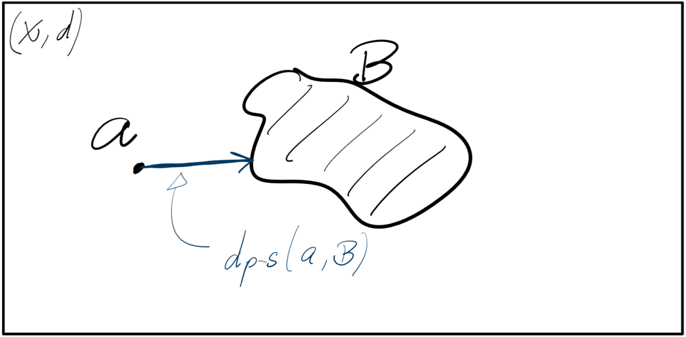

Figure: Point-to-set distance $d_{\mathrm{p-S}}(a, B)$.

Set-to-set distance $d_{\mathrm{S-S}}$

...then measure the distance from a set to a set...

Definition

Given a metric space (X, d) and $A, B \in \mathcal{H}(X)$,
The *set-to-set distance* from A to B is
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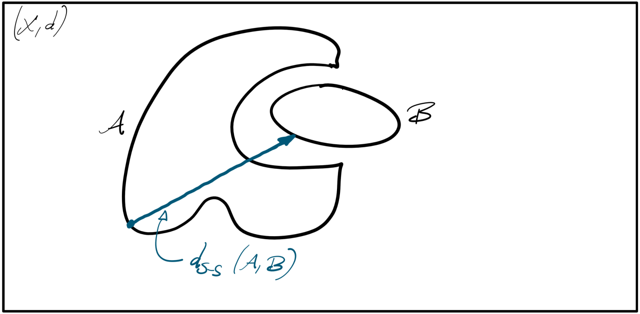

Figure: Set-to-set distance $d_{\mathrm{S-S}}(A, B)$

Is $d_{\mathrm{S-S}}$ a metric?

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$d_{\mathrm{S-S}}(A, B) \geq 0$: clear.

$d_{\mathrm{S-S}}(A, B) = 0$ iff. $A = B$ might not be true...

$d_{\mathrm{S-S}}(A, B) = d_{\mathrm{S-S}}(B, A)$ might not be true...

$d_{\mathrm{S-S}}(A, B) \leq d_{\mathrm{S-S}}(A, C) + d_{\mathrm{S-S}}(C, B)$: takes work; is true.

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Consider...

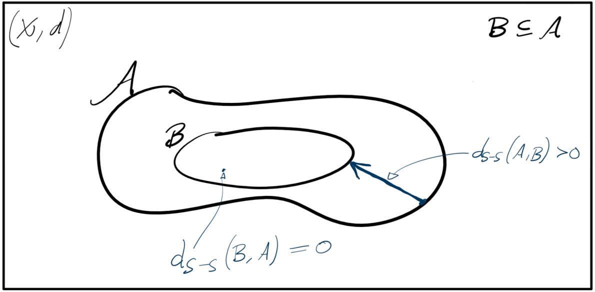

Figure: $d_{\mathrm{S-S}}$ is not a metric on $\mathcal{H}(X)$.

Pompeiu-Hausdorff metric $d_{\mathcal{H}(X)}$

We fix these problems by forcing our ‘distance’ to take the worst-case scenario.

Definition

Given a metric space (X, d),
The *Pompeiu-Hausdorff metric* on $\mathcal{H}(X)$ is the function

$$\begin{aligned} d_{\mathcal{H}(X)} : \mathcal{H}(X) \times \mathcal{H}(X) &\longrightarrow \mathbb{R}_{\geq 0} \\ (A, B) &\longmapsto \max\{d_{\mathrm{S-S}}(A, B), d_{\mathrm{S-S}}(B, A)\} \end{aligned}$$

$d_{\mathcal{H}(X)}(A, B) < \varepsilon$ is equivalent to $d_{\mathrm{S-S}}(A, B) < \varepsilon$ and $d_{\mathrm{S-S}}(B, A) < \varepsilon$.

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$d_{\mathcal{H}(X)}(A, B) < \varepsilon$ is equivalent to $d_{\mathrm{S-S}}(A, B) < \varepsilon$ and $d_{\mathrm{S-S}}(B, A) < \varepsilon$.
Remark. The $\inf$ and $\sup$ seen so far are $\min$ and $\max$ when we restrict to the *compact* sets in $\mathcal{H}(X)$.

$(\mathcal{H}(X), d_{\mathcal{H}(X)})$ is a metric space

Proposition

Given a metric space (X, d),
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Proof sketch.
Let $A, B, C \in \mathcal{H}(X)$.

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($d_{\mathcal{H}(X)}(A, B) = 0 \implies A = B$):
Since $0 = d_{\mathrm{S-S}}(A, B) = \sup_{a \in A}\{d_{\mathrm{p-S}}(a, B)\}$, we have that for all $a \in A$, $0 = d_{\mathrm{p-S}}(a, B) = \inf_{b \in B}\{d(a, b)\}$. Hence, a is a limit point of B; thus, $A \subseteq \bar{B}$. Symmetrically, $B \subseteq \bar{A}$.

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A, B are compact subsets of a metric space, so they are closed. Hence, $A = \bar{A} = \bar{B} = B$.

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A, B are compact subsets of a metric space, so they are closed. Hence, $A = \bar{A} = \bar{B} = B$.
($d_{\mathcal{H}(X)}(A, B) = d_{\mathcal{H}(X)}(B, A)$): Obviously clear.

$(\mathcal{H}(X), d_{\mathcal{H}(X)})$ is a metric space

Proof sketch. (cont.)
($d_{\mathcal{H}(X)}(A,B) \leq d_{\mathcal{H}(X)}(A,C) + d_{\mathcal{H}(X)}(C,B)$):
For all $a \in A, b \in B, c \in C$, we have that $d(a,b) \leq d(a,c) + d(c,b)$. The triangle inequality for $d_{\mathrm{S-S}}$ is obtained by taking appropriate infima/suprema. This then gives the triangle inequality for $d_{\mathcal{H}(X)}$.

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Completeness of $(\mathcal{H}(X), d_{\mathcal{H}(X)})$

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Given a metric space (X, d),
(X, d) is *complete* if for every Cauchy sequence $(x_n)_{n=1}^{\infty}$ in X, $(x_n)_{n=1}^{\infty}$ converges.

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If (X, d) is complete, then so is $(\mathcal{H}(X), d_{\mathcal{H}(X)})$.

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Let $(A_n)_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{H}(X)$. What could the limit be?

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Let $(A_n)_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{H}(X)$. What could the limit be?
Set $B = \{x \in X \mid x$ is a limit point of some $(a_n)_{n=1}^{\infty}$ with each $a_n \in A_n\}$ and $A = \bar{B}$.

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Proof sketch. (cont.)
Reminder:
$B = \{x \in X \mid x$ is a limit point of some $(a_n)_{n=1}^{\infty}$ with each $a_n \in A_n\}$ and $A = \bar{B}$.
(A is non-empty): Take $(N_k)_{k=1}^{\infty}$ a strictly increasing sequence of positive integers such that for all $m, n \geq N_k$, $d_{\mathcal{H}(X)}(A_m, A_n) < 2^{-k}$.

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Some work shows that we can take a sequence $(a_n)_{n=1}^{\infty}$ with each $a_n \in A_n$ and $d(a_{N_i}, a_{N_j})$ gets arbitrarily small when $i, j \geq k$ for large enough k.

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$B = \{x \in X \mid x$ is a limit point of some $(a_n)_{n=1}^{\infty}$ with each $a_n \in A_n\}$ and $A = \bar{B}$.
(A is non-empty): Take $(N_k)_{k=1}^{\infty}$ a strictly increasing sequence of positive integers such that for all $m, n \geq N_k$, $d_{\mathcal{H}(X)}(A_m, A_n) < 2^{-k}$.
Some work shows that we can take a sequence $(a_n)_{n=1}^{\infty}$ with each $a_n \in A_n$ and $d(a_{N_i}, a_{N_j})$ gets arbitrarily small when $i, j \geq k$ for large enough k.
Hence, the subsequence $(a_{N_k})_{k=1}^{\infty}$ is Cauchy. As (X, d) is complete, $(a_{N_k})_{k=1}^{\infty}$ converges to some x. By definition, $x \in B$, so $x \in A$ and A is non-empty.

Completeness of $(\mathcal{H}(X), d_{\mathcal{H}(X)})$

Proof sketch. (cont.)
Reminder:
$B = \{x \in X \mid x$ is a limit point of some $(a_n)_{n=1}^{\infty}$ with each $a_n \in A_n\}$ and $A = \bar{B}$.
($(A_n)_{n=1}^{\infty}$ converges to A): Need to show that $\lim_{n\to\infty} d_{\mathcal{H}(X)}(A_n, A) = 0$.
i.e. need to show $\lim_{n\to\infty} d_{\mathrm{S-S}}(A_n, A) = \lim_{n\to\infty} d_{\mathrm{S-S}}(A, A_n) = 0$.

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i.e. need to show $\lim_{n\to\infty} d_{\mathrm{S-S}}(A_n, A) = \lim_{n\to\infty} d_{\mathrm{S-S}}(A, A_n) = 0$.
$\lim_{n\to\infty} d_{\mathrm{S-S}}(B, A_n) = \lim_{n\to\infty} d_{\mathrm{S-S}}(A_n, B) = 0$ can be shown with a similar sequence argument to showing $A \neq \varnothing$. Hence, $\lim_{n\to\infty} d_{\mathcal{H}(X)}(A_n, B) = 0$. Since $d_{\mathcal{H}(X)}(A, B) = 0$ (won't prove this; holds for closures in general), we have that $\lim_{n\to\infty} d_{\mathcal{H}(X)}(A_n, A) = 0$.

Completeness of $(\mathcal{H}(X), d_{\mathcal{H}(X)})$

Proof sketch. (cont.)
Reminder:
$B = \{x \in X \mid x$ is a limit point of some $(a_n)_{n=1}^{\infty}$ with each $a_n \in A_n\}$ and $A = \bar{B}$.
(A is compact):
We leverage the fact that compact $\iff$ complete and totally bounded in metric spaces.

Completeness of $(\mathcal{H}(X), d_{\mathcal{H}(X)})$

Proof sketch. (cont.)
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Claim. A is closed.
$A = \bar{B}$ is a closure.

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Proof sketch. (cont.)
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We leverage the fact that compact $\iff$ complete and totally bounded in metric spaces.
Claim. A is closed.
$A = \bar{B}$ is a closure.
Claim. A is complete.
A is a closed subset of a complete metric space. Thus, A is complete.

Completeness of $(\mathcal{H}(X), d_{\mathcal{H}(X)})$

Proof sketch. (cont.)
Reminder:
$B = \{x \in X \mid x$ is a limit point of some $(a_n)_{n=1}^{\infty}$ with each $a_n \in A_n\}$ and $A = \bar{B}$.
Claim. A is totally bounded.
Fix $\varepsilon > 0$ and take $n \in \mathbb{Z}_{>0}$ large enough so that $d_{\mathcal{H}(X)}(A_n, A) < \varepsilon/2$.
Since A_n is compact, it is totally bounded and hence there are finitely many $a_1, \ldots, a_k$ such that $A_n \subseteq \bigcup_{i=1}^{k} \left(B_{\varepsilon/2}(a_i)\right)$.

Completeness of $(\mathcal{H}(X), d_{\mathcal{H}(X)})$

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Take $x \in X$. Since $d_{\mathcal{H}(X)}(A, A_n) < \varepsilon/2$, there is $a \in A_n$ such that $d(x, a) < \varepsilon/2$. Also, for some i, $d(a, a_i) < \varepsilon/2$. Thus, $d(x, a_i) < \varepsilon$, so $x \in \bigcup_{i=1}^{k} (B_{\varepsilon}(a_i))$. Hence, A is totally bounded.

Completeness of $(\mathcal{H}(X), d_{\mathcal{H}(X)})$

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Claim. A is compact.

A is a complete and totally bounded subset of a metric space, so A is compact.

Completeness of $(\mathcal{H}(X), d_{\mathcal{H}(X)})$

Proof sketch. (cont.)
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$B = \{x \in X \mid x$ is a limit point of some $(a_n)_{n=1}^{\infty}$ with each $a_n \in A_n\}$ and $A = \bar{B}$.

Claim. A is totally bounded.
Fix $\varepsilon > 0$ and take $n \in \mathbb{Z}_{>0}$ large enough so that $d_{\mathcal{H}(X)}(A_n, A) < \varepsilon/2$. Since A_n is compact, it is totally bounded and hence there are finitely many $a_1, \dots, a_k$ such that $A_n \subseteq \bigcup_{i=1}^{k} \left(B_{\varepsilon/2}(a_i)\right)$.
Take $x \in X$. Since $d_{\mathcal{H}(X)}(A, A_n) < \varepsilon/2$, there is $a \in A_n$ such that $d(x, a) < \varepsilon/2$. Also, for some i, $d(a, a_i) < \varepsilon/2$. Thus, $d(x, a_i) < \varepsilon$, so $x \in \bigcup_{i=1}^{k} (B_\varepsilon(a_i))$. Hence, A is totally bounded.

Claim. A is compact.
A is a complete and totally bounded subset of a metric space, so A is compact.
Thus, $A \in \mathcal{H}(X)$ and $(A_n)_{n=1}^{\infty}$ converges to A.

Completeness of $(\mathcal{H}(X), d_{\mathcal{H}(X)})$

Remark. An alternative version of this proof is an exercise in Munkres' *Topology* (Second Edition; Chapter 45, Page 280, Exercise 7). Munkres adapts a different, but equivalent definition of $d_{\mathcal{H}(X)}$. Showing the equivalence is also a good exercise.

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We now have all the tools we need to show that a self-similar shape exists.

Outline

1. Goals
2. Motivation (Pretty Pictures)
3. Iterated Function Systems and Self-Similarity
4. The Hausdorff Metric
5. Existence of Self-Similar Shapes
6. Aside: Fractals

Definitions from a while ago

Recall Definitions

Given a metric space (X, d),
An *IFS* $\Psi = \{\psi_1, \dots, \psi_k\}$ on X is a set of contraction maps $\psi_i : X \to X$.
The *iteration map* is the function

$$\begin{aligned}\Psi^1 : \mathcal{H}(X) &\longrightarrow \mathcal{H}(X)\\ A &\longmapsto \bigcup_{\psi \in \Psi} (\psi(A))\end{aligned}$$

A subset $A \in \mathcal{H}(X)$ is called *self-similar* if there exists an IFS Ψ on X such that $A = \Psi^1(A)$.

We usually take (X, d) complete, so then $(\mathcal{H}(X), d_{\mathcal{H}}(X))$ is complete.
We're interested in a *fixed point* of a map on a *complete metric space*.
What gives us information about this?

Contraction Mapping Theorem

Theorem *(Contraction Mapping)*

Given (X, d) a metric space and $f : X \to X$,
If f is a contraction and (X, d) is complete, then f has a unique fixed point (a point $x \in X$ such that $f(x) = x$).

Proof. MATH2401. Very pretty.

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Proof. MATH2401. Very pretty.

If we can show that the iteration map Ψ^1 is contractive, this will show that *any IFS on a complete metric space has a unique associated self-similar set*.

The iteration map is a contraction

Lemma

Given a metric space (X, d) and an IFS Ψ on X,
The iteration map Ψ^1 indeed maps elements of $\mathcal{H}(X)$ to elements of $\mathcal{H}(X)$, and Ψ^1 is a contraction map on $(\mathcal{H}(X), d_{\mathcal{H}(X)})$.

Proof.

(Ψ^1 maps elements of $\mathcal{H}(X)$ to elements of $\mathcal{H}(X)$):

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(Ψ^1 maps elements of $\mathcal{H}(X)$ to elements of $\mathcal{H}(X)$):
Let $A \in \mathcal{H}(X)$. Then, for each $\psi \in \Psi$, since ψ is a contraction, it is continuous (easy to verify). Since A is compact, it follows that $\psi(A)$ is compact.

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(Ψ^1 maps elements of $\mathcal{H}(X)$ to elements of $\mathcal{H}(X)$):
Let $A \in \mathcal{H}(X)$. Then, for each $\psi \in \Psi$, since ψ is a contraction, it is continuous (easy to verify). Since A is compact, it follows that $\psi(A)$ is compact.
Because $\Psi^1(A) = \bigcup_{\psi \in \Psi}(\psi(A))$ is a finite union of compact sets, $\Psi^1(A)$ is compact. Thus, $\Psi^1(A) \in \mathcal{H}(X)$.

The iteration map is a contraction

Proof. (cont.)
(Ψ^1 is a contraction):
Let $A, B \in \mathcal{H}(X)$. Then,

$$
\begin{aligned}
d_{\mathrm{S-S}}(\Psi^1(A), \Psi^1(B)) &= \sup_{x \in \Psi^1(A)} \left\{ \inf_{y \in \Psi^1(B)} \{d(x,y)\} \right\} \\
&= \sup_{x \in \bigcup_{\psi \in \Psi} (\psi(A))} \left\{ \inf_{y \in \bigcup_{\psi' \in \Psi} (\psi'(B))} \{d(x,y)\} \right\} \\
&= \max_{\psi \in \Psi} \left\{ \sup_{x \in \psi(A)} \left\{ \min_{\psi' \in \Psi} \left\{ \inf_{y \in \psi'(B)} \{d(x,y)\} \right\} \right\} \right\}
\end{aligned}
$$

$$
d_{\mathrm{S-S}}(\Psi^1(A), \Psi^1(B)) = \max_{\psi \in \Psi} \left\{ \sup_{a \in A} \left\{ \min_{\psi' \in \Psi} \left\{ \inf_{b \in B} \{d(\psi(a), \psi'(b))\} \right\} \right\} \right\}
$$

The iteration map is a contraction

Proof. (cont.)

$$
\begin{aligned}
d_{\mathrm{S-S}}(\Psi^1(A), \Psi^1(B)) &= \max_{\psi\in\Psi}\left\{\sup_{a\in A}\left\{\min_{\psi'\in\Psi}\left\{\inf_{b\in B}\{d(\psi(a), \psi'(b))\}\right\}\right\}\right\} \\
&\leq \max_{\psi\in\Psi}\left\{\sup_{a\in A}\left\{\inf_{b\in B}\{d(\psi(a), \psi(b))\}\right\}\right\} \\
&\leq \max_{\psi\in\Psi}\left\{\sup_{a\in A}\left\{\inf_{b\in B}\{\mathrm{Lip}(\psi)\, d(a, b)\}\right\}\right\} \\
&= \max_{\psi\in\Psi}\{\mathrm{Lip}(\psi)\} \sup_{a\in A}\left\{\inf_{b\in B}\{d(a, b)\}\right\} \\
&= \max_{\psi\in\Psi}\{\mathrm{Lip}(\psi)\}\, d_{\mathrm{S-S}}(A, B) \\
\implies d_{\mathrm{S-S}}(\Psi^1(A), \Psi^1(B)) &\leq \max_{\psi\in\Psi}\{\mathrm{Lip}(\psi)\}\, d_{\mathcal{H}(X)}(A, B)
\end{aligned}
$$

Since each $\psi \in \Psi$ is a contraction and there are finitely many $\psi \in \Psi$, we have that $\max_{\psi\in\Psi}\{\mathrm{Lip}(\psi)\} < 1$. Therefore, Ψ^1 is contractive.

The point of this talk

So finally...

Corollary

Given a complete metric space (X, d),
Each IFS Ψ on X admits a unique self-similar set.
Having already seen an IFS in $\mathbb{R}^2$, this guarantees that a self-similar set exists.

Proof.

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Proof.
Ψ^1 is a contraction.

Remark. The unique self-similar set Ψ admits is known as the *attractor* of Ψ (hence the name of this talk).

The point of this talk: Example

Examples

The self-similar set I passed around the room (affectionately, my ‘Sierpinski pyramid’) has the IFS $\Psi = \{\psi_1, \ldots, \psi_6\}$ for $\psi_i : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\psi_1 : x \longmapsto \frac{1}{2}x \qquad \psi_2 : x \longmapsto \frac{1}{2}x + \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}$$

$$\psi_3 : x \longmapsto \frac{1}{2}x + \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix} \qquad \psi_4 : x \longmapsto \frac{1}{2}x + \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

$$\psi_5 : x \longmapsto \frac{1}{2}x + \begin{pmatrix} 1/4 \\ 1/4 \\ \sqrt{7/2} \end{pmatrix} \qquad \psi_6 : x \longmapsto -\frac{1}{2}x + \begin{pmatrix} 1/4 \\ 1/4 \\ \sqrt{7/2} \end{pmatrix}$$

Remark. That’s my favourite fractal. Speaking of fractals...

Outline

1 Goals

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3 Iterated Function Systems and Self-Similarity

4 The Hausdorff Metric

5 Existence of Self-Similar Shapes

6 Aside: Fractals

Fractals

Definition

Given an appropriate space X,
A subset A of X is called a *fractal* if it has non-integer dimension.
...Or other definitions, depending who you ask (Mandelbrot: Hausdorff dimension $>$ topological dimension).

The name comes from Latin *Fractus*, roughly meaning 'broken'
The notion of dimension must be appropriately taken in context. So must the requirements of the space X ($X = \mathbb{R}^n$ is common).

Self-similarity dimension of the Sierpinski Triangle

Examples

Consider the Sierpinski Triangle $\mathcal{S}$. Scaling $\mathcal{S}$ by $1/2$ (towards the bottom-left corner) reduces the 'size'* of $\mathcal{S}$ by $1/3$, so the dimension d satisfies $(1/2)^d = 1/3$. Hence, $\mathcal{S}$ has self-similarity dimension $d = \log_2(3)$.

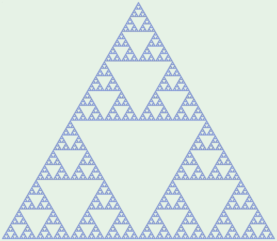

Figure: Sierpinski Triangle $\mathcal{S}$

*Measure-theoretic details swept *way* under the rug. Relevant concepts: *Hausdorff measure* $\mathcal{H}^d$, *Hausdorff dimension* $\dim_{\mathcal{H}}$.

Connection between fractals and self-similarity

Question: Are all self-similar shapes fractals?

Connection between fractals and self-similarity

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No. Here's the IFS of a square: $\Psi = \{\psi_1, \ldots, \psi_4\}$ with $\psi_i : \mathbb{R}^2 \to \mathbb{R}^2$ given by

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Question: Are all fractals self-similar?
No. The west coast of Great Britain is a fractal with dimension ≈ 1.25 (src: Wikipedia: Coastline paradox), but Great Britain doesn't contain another Great Britain.

Connection between fractals and self-similarity

"Fractals are typically not self-similar" (Grant Sanderson; 3B1B).

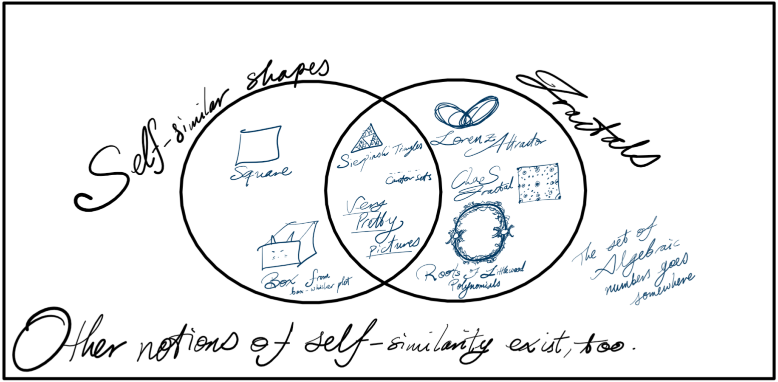

Figure: Self-Similarity compared to Fractals

Roots of Littlewood Polynomials (beauty.pdf)

Chaotic Sensing (ChaoS) fractal

Thanks for listening!

I hope you have a better understanding and appreciation of self-similarity. Any feedback on my talk would be very helpful.