Chaotic Systems

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Presentation outline

- Review of dynamical systems
- 'Definitions' of Chaos
  - Sensitivity of intitial conditions
  - Topological transitivity
  - Denseness of periodicity
- The three-body problem and Poincaré
- Lorenz system and strange attractors
- Bonus Applications (time dependent)

Review of dynamical systems

A system where the next state is dependent on the current state.
Suppose $\boldsymbol{x}$ is an element of some space T, commonly $\mathbb{R}^n$

- Discrete case: $\boldsymbol{x}_{n+1} = f(\boldsymbol{x}_n, n), n \in \mathbb{N}$.
- Continuous case: $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, t), t \in \mathbb{R}$.

where $\boldsymbol{x} \in \Omega, t \in \mathbb{R}$.
A system is called autonomous if there is no time dependency i.e.
$\dot{\boldsymbol{x}} = f(\boldsymbol{x})$ or $\boldsymbol{x}_n + 1 = f(\boldsymbol{x}_n)$.

Example: Arnold's Cat Map

Suppose we have an image of size $N \times N$ comprised of discrete points (in this case coloured pixels). Each points's current position can be denoted as (x_n, y_n).
Arnold's Cat Map is the discrte dynamical system where,

$$x_{n+1} = (2x_n + y_n) \mod N$$
$$y_{n+1} = (x_n + y_n) \mod N$$

This represents a stretching of the image and then compression back into the original shape.

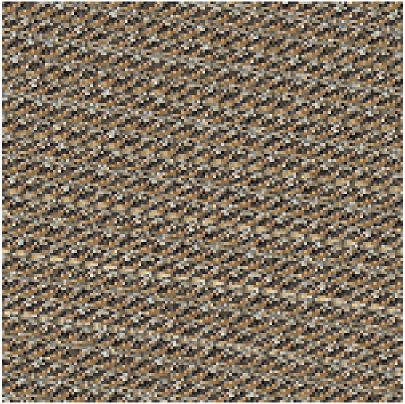
Iterations: 0

Iterations: 1

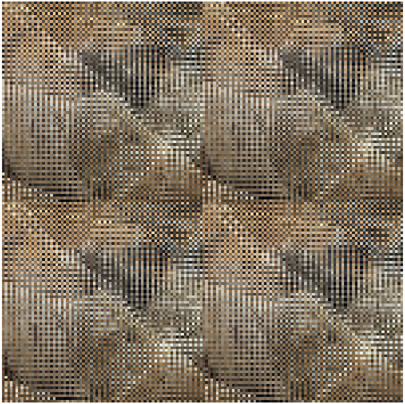
Iterations: 2

Iterations: 10

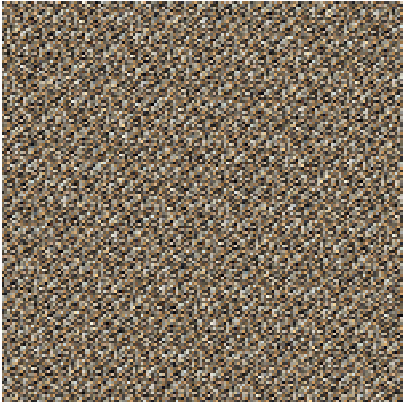
Iterations: 53



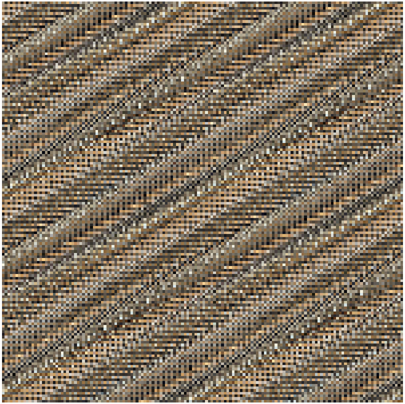
Iterations: 100



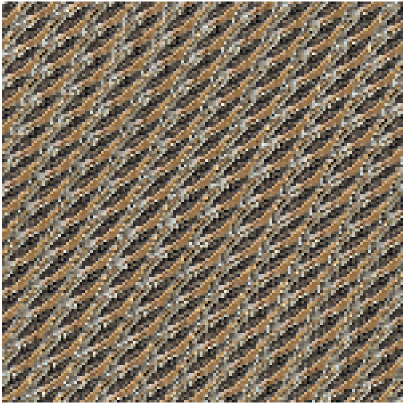
Iterations: 127



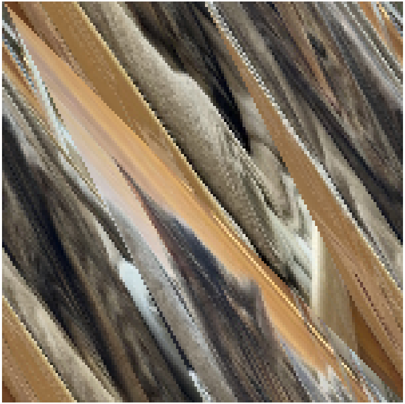
Iterations: 202



Iterations: 290



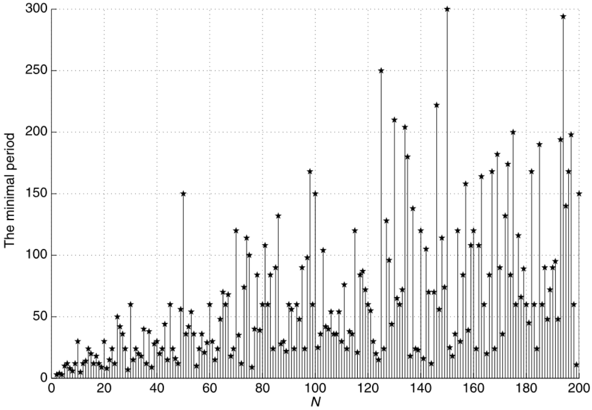
Iterations: 299



Iterations: 300

Periodicity

Although 300 iterations seems like a nice number to have for a period (especially as the image is 150 pixels wide or $N = 150$) the actual periodicity of this map is more complex.



Chaotic systems

Although conceptually somewhat intuitive, there is no unified mathematical definition for what chaos is.

- "The complicated aperiodic attracting orbits of certain, usually low-dimensional dynamical systems" - Phillip Holmes
- "A kind of order without periodicity" - Bai-Lin Hao
- "When the present determines the future, but the approximate present does not approximately determine the future" - Edward Lorenz

Chaotic systems

More mathematically, a system can be *called* chaotic if it,

- is sensitive to the initial conditions of the system
- is topologically transitive
- has a dense collection of points with periodic orbits

Initial condition sensitivity and Lyapunov exponents

We can characterise the sensitivity of a system to intial conditions using Lyapunov exponents.
Suppose we have two trajectories of a dynamic system, $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ which are seperated by a vector $\boldsymbol{\delta}(t)$ such that $\boldsymbol{y}(t) = \boldsymbol{x}(t) + \boldsymbol{\delta}(t), \forall t \in \mathbb{R}$.
If $\boldsymbol{x}$ and $\boldsymbol{y}$ begin infinitesimally close (i.e. $||\boldsymbol{\delta}(0)|| = \varepsilon > 0$) then we can say that $||\boldsymbol{\delta}(t)|| \approx \varepsilon e^{\lambda t}$ for some λ called the Lyapunov exponent.
A positive maximum Lyapunov exponent is *generally* an indicator of a chaotic system

Topological transitivity

A map $f : X \to X$ is topologically transitive if for any pair of non-empty open sets $U, V \subset X$ there exists a $k > 0$ such that $f^k(U) \cap V \neq \emptyset$.
In words, repeated applications of a function on any subset of its domain eventually overlaps with any other subset.
Or, the function effectively mixes a subset of points across the domain.
This condition on chaotic systems implies that it is impossible to decompose the system into two open sets.

Denseness of periodic orbits

Given any point there is a periodic orbit that comes arbitrarily close to it.
Counterintuitive?
Due to the sensitivity to initial conditions and topological transitivity these periodic orbits must be unstable, thus guaranteeing an abundance of aperiodic behaviour.

The three body problem

Three bodies with masses m_1, m_2, m_3 and positions $\mathbf{r_1}, \mathbf{r_2}, \mathbf{r_3}$ interact with each other via gravity in space.
Using Newton's laws of motion and gravity we can generate the following sets of second order differential equations.

$$\ddot{\mathbf{r}}_1 = -Gm_2 \frac{\mathbf{r_1} - \mathbf{r_2}}{|\mathbf{r_1} - \mathbf{r_2}|^3} - Gm_3 \frac{\mathbf{r_1} - \mathbf{r_3}}{|\mathbf{r_1} - \mathbf{r_3}|^3},$$
$$\ddot{\mathbf{r}}_2 = -Gm_1 \frac{\mathbf{r_2} - \mathbf{r_1}}{|\mathbf{r_2} - \mathbf{r_1}|^3} - Gm_3 \frac{\mathbf{r_2} - \mathbf{r_3}}{|\mathbf{r_2} - \mathbf{r_3}|^3},$$
$$\ddot{\mathbf{r}}_3 = -Gm_1 \frac{\mathbf{r_3} - \mathbf{r_1}}{|\mathbf{r_3} - \mathbf{r_1}|^3} - Gm_2 \frac{\mathbf{r_3} - \mathbf{r_2}}{|\mathbf{r_3} - \mathbf{r_2}|^3},$$

Poincaré

In the 1880s, French mathematician Henri Poincaré became interested in solving this problem, by first analysing the *restricted* three body problem.
Poincaré's plan was to first solve this version and then generalise the solution into the full three body problem.

Poincaré

Poincaré discovered that for this restricted case, "*the three bodies will return arbitrarily close to their initial position infinitely many times*"

However when he attempted to generalise he discovered that most orbits neither converged to a periodic orbit or a fixed point but yet remained bounded.

He concluded that he did not have the analytic tools required to solve the general case.

Poincaré

Poincaré was deeply disheartened with these results, and wrote

> *I believed when I started this work that once the solution of the problem was found for the specific case that I dealt with it would be immediately generalizable without having to overcome any new difficulties outside of those which are due to the larger number of variables and the impossibility of a geometric representation. I was mistaken.*

The three body problem

We now know that the three body problem does not have a closed form solution (i.e. one which can be expressed with a finite number of mathematical solutions).

However, of more interest to us are the orbits Poincaré was frustrated by.

What Poincaré discovered by accident was the first evidence of chaos in a well defined dynamic system.

It is worth noting how ahead of its time this discovery is.

Lorenz, Fetter, and Hamilton

The next breakthrough in chaos theory was 70 years after Poincaré's discovereries and 16 years after the invention of the computer.

While investigating a simple atmospheric model mathematician Edward Lorenz, and computer scientists Ellen Fetter and Margaret Hamilton attempted to rerun the second half of a simulation.

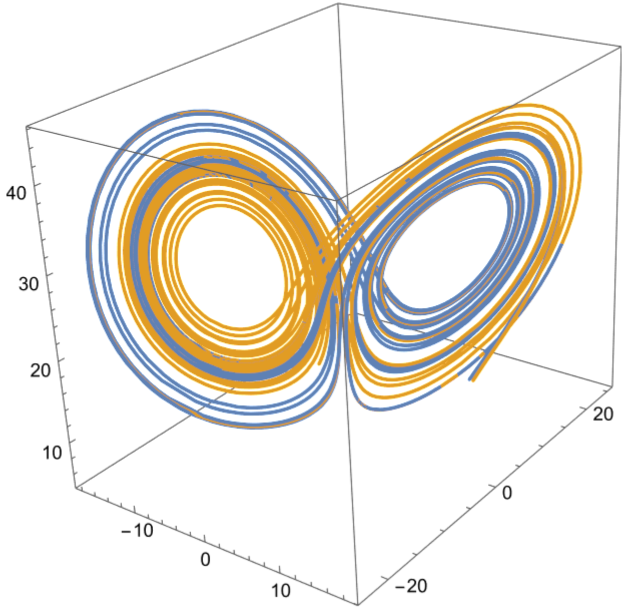
After entering the point's position at the half time point, they discovered the restarted simulation was wildly different from the original.

The Lorenz system

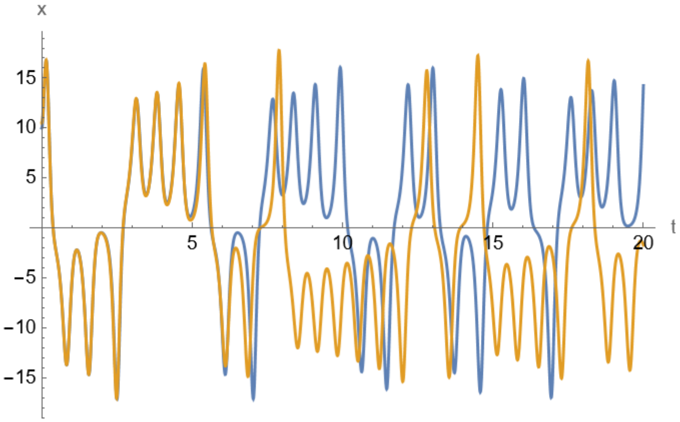
$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z \end{aligned}$$

where σ, ρ, β are constant parameters.

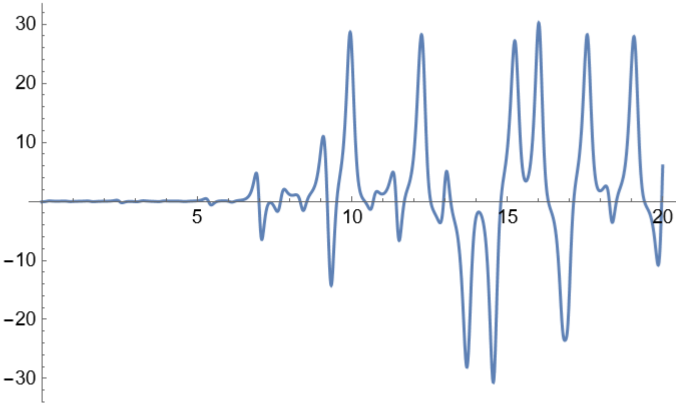
The Lorenz system



The Lorenz system



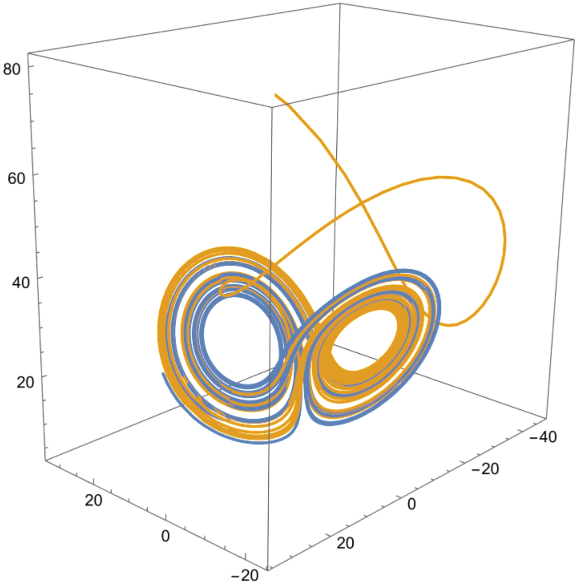
The Lorenz system



The Lorenz system

The shape of a particles trajectory, inspired Lorenz to coin the term the 'Butterfly effect' to describe the sensitivity of the system to initial conditions.
Additionally this system demonstrates a 'strange attractor'

The Lorenz system



The Lorenz system

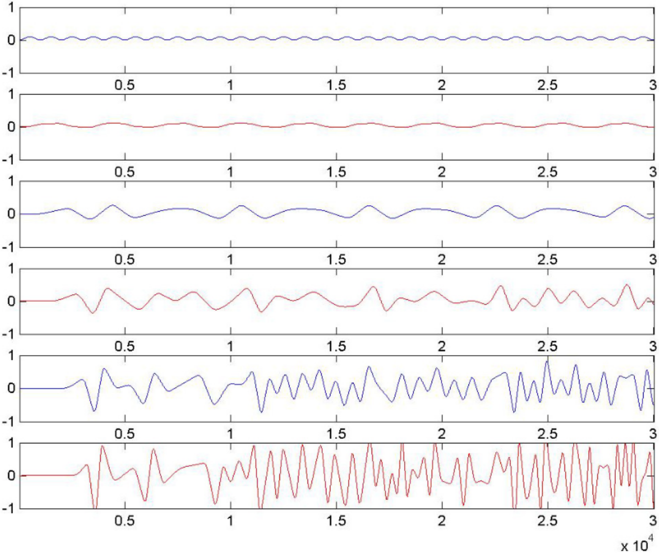
Ultimately these results frustrated Lorenz who stated,

> *. . . that [the] prediction of the sufficiently distant future is impossible by any method, unless the present conditions are known exactly.*

Traffic Modelling

Many traffic models assume things like a steady density of cars, and a steady flow.
When these assumptions are discarded, models can exhibit chaotic behaviour.

Traffic Modelling



Traffic Modelling

Looking at the Lyapunov exponents of each car,

Car	Lyapunov exponent
1	-0.2873
2	-0.1367
3	-0.0039
4	2.8287
5	3.0316
6	4.5879

Clear to see that cars 4-6 exhibit chaotic behaviour.

Traffic Modelling

This type of chaos analysis can also be applied to entire traffic system, as oppposed to car by car basis, and be used to predict what events cause chaos and to design systems to avoid chaotic behaviour.

Population Modelling

$$\frac{dX}{dt} = a_1 X - b_1 X^2 - \frac{wYX}{X + D},$$
$$\frac{dY}{dt} = -a_2 Y + \frac{w_1 YX}{X + D_1} - \frac{w_2 YU}{Y + D_2},$$
$$\frac{dZ}{dt} = AZ(1 - \frac{Z}{K}) - \frac{w_3 UZ}{Z + D_3},$$
$$\frac{dU}{dt} = cU - \frac{w_4 U^2}{Y + Z}$$

Where species X is a prey animal, Y is a specialist predator which only eats X, Z is another prey animal, and U is a generalist predator.

Population Modelling

Authors found that with certain parameters this system behaves chaotically and is hard to predict.
Authors also found that this is especially the case after the loss of one species.

Questions