Supermanifolds - Who Cares?

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Maths Talks Week 5

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Motivation For Supergeometry

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Bosons require commuting operators while fermions require anti-commuting operators.

Locally, a supermanifold has both even and odd coordinates.

History of the Theory

Mathematician	Year	Type of Supermanifold
Berezin	1987	Topological Manifold + Sheaf
DeWitt	1984	Set + Atlas
Leites	1980	Topological Manifold + Sheaf
Batchelor	1980	Set + Atlas
Rogers	1980	Set + Atlas
Kostant	1975	Topological Manifold + Sheaf

The goal of this project was to understand each definition and their connections.

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$$X = X_0 + \sum_{i=1}^{k} X_i \beta_i + \sum_{i<j} X_{ij} \beta_i \beta_j + \cdots +$$
$$+ \sum_{\mu_1 < \cdots < \mu_{k-1}} X_{\mu_1 \cdots \mu_{k-1}} \beta_{\mu_1} \cdots \beta_{\mu_{k-1}} + X_{1 \cdots k} \beta_1 \cdots \beta_k.$$

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Example

Take $X \in B_2$. X could be $1 + 2\beta_1 + 3\beta_2 + 4\beta_1\beta_2$.

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- DeWitt's supermanifold uses W_∞ (a Grassman algebra with infinite generators.)
- Batchelor's supermanifold uses B_n but places a coarse topology on it.

Flat Superspace and the Body Map

Define flat superspace as $B_k^{m,n} := \underbrace{B_{k,0} \times \cdots \times B_{k,0}}_{\text{m times}} \times \underbrace{B_{k,1} \times \cdots \times B_{k,1}}_{\text{n times}}.$

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Take a supernumber $X = 1 + 2\beta_1 + 3\beta_2 + 4\beta_1\beta_2$ in B_2.

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The body map extends to $\varepsilon_{m,n} : B_k^{m,n} \to \mathbb{R}^m$ by
$(x; \xi) \mapsto (\varepsilon(x_1), \cdots, \varepsilon(x_m))$.

Topology

We can place two interesting topologies on superspace:

- $U \subset B_L^{m,n}$ is open in the DeWitt topology if there is an open set $V \subset \mathbb{R}^m$ such that $\varepsilon_{m,n}^{-1}(V) = U$.

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The DeWitt topology is not even Hausdorff! Yet somehow it still makes the most sense on these spaces...

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We say f is G^∞ if there exist smooth $f_\mu : \varepsilon_{m,n}(U) \to B_k$ such that

$$f(x;\xi) = \sum_\mu \widehat{f_\mu}(x;\xi)\xi_\mu.$$

Here $\mu = (\mu_1, \cdots, \mu_l)$ such that $1 \le \mu_1 < \cdots < \mu_l \le k$.

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Define $f : B_2^{1,1} \to B_2$ by $(x, \xi) \mapsto (1 + 2\beta_1 + 3\beta_2 + 4\beta_1\beta_2) + \xi$.

Define $f_0(\varepsilon(x)) = 1 + 2\beta_1 + 3\beta_2 + 4\beta_1\beta_2, \quad f_1(\varepsilon(x)) = 1$.

Extending the domain of f_0 and f_1, we find

$$f(x; \xi) = \widehat{f_0}(x, \xi) + \widehat{f_1}(x, \xi)\xi$$

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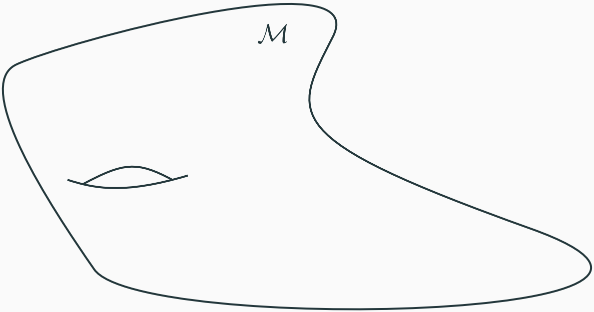
We can also have an H^∞ function, where the f_μ map into $\mathbb{R}$ not B_k.

Smooth Manifolds



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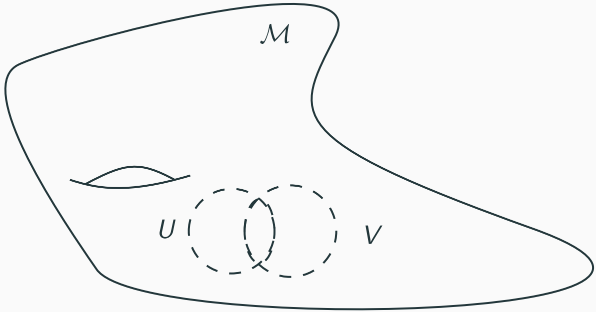

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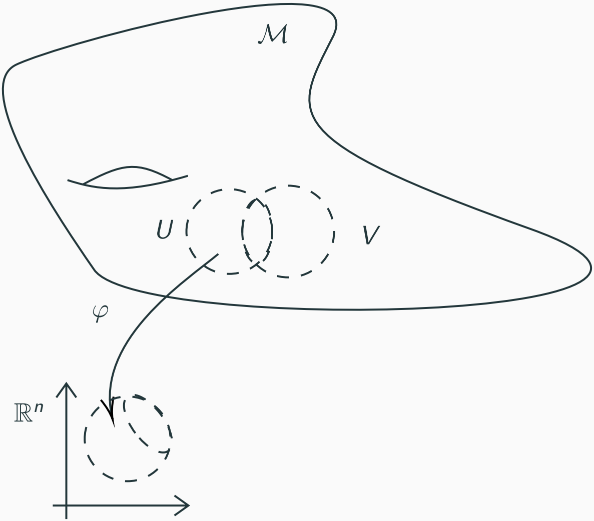

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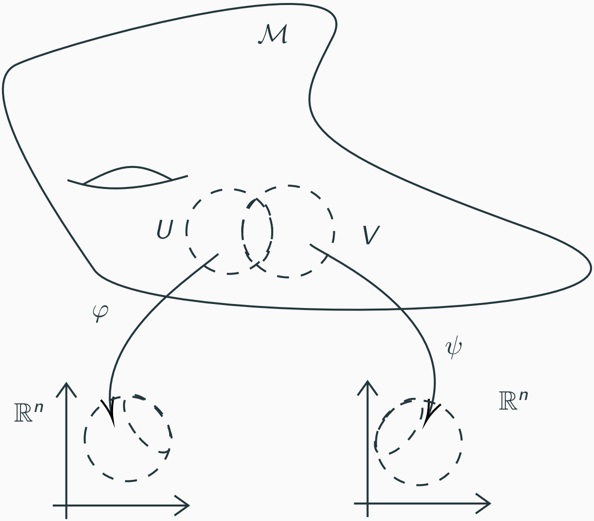

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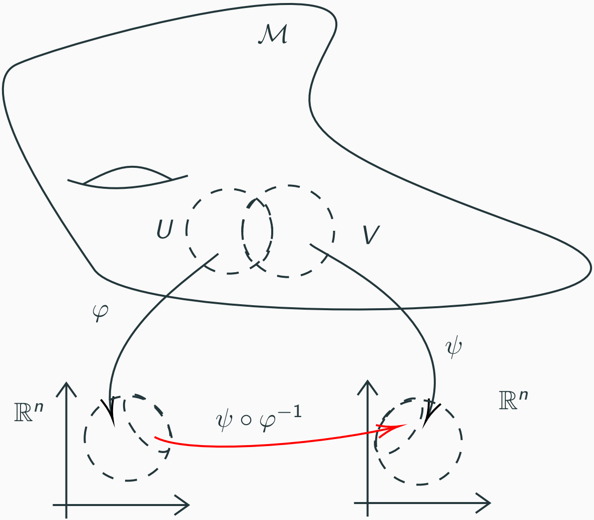

G^∞ Supermanifolds - Rogers' Approach

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- Lie supergroups - general linear supergroup $GL(m|n)$, special linear supergroup $SL(m|n)$, orthosymplectic supergroup $OSP(m|n)$.

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The above defines a sheaf!

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$\mathcal{M} = (M, \mathcal{O}_M)$ is a Berezin-Kostant-Leites supermanifold of dimension p, q if $\mathcal{O}_M$ is locally isomorphic to $\mathcal{C}^{\infty}_{\mathbb{R}^p} \otimes \bigwedge(\xi_1, \cdots, \xi_q)$ and M is locally homeomorphic to $\mathbb{R}^p$.

Theorem

The definition of Berezin-Kostant-Leites supermanifolds is equivalent to supermanifolds that use B_k (finite generators), the DeWitt topology (coarse, non-hausdorff), and H^{∞} functions.

Tying the Definitions Together