How Big Do Numbers Get?

William Barnett

Inspiration

The trouble with integers is that we have examined only the very small ones. Maybe all the exciting stuff happens at really big numbers, ones we can't even begin to think about in any very definite way.
(Ronald Graham)

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(Ronald Graham)

It wasn't infinity in fact. Infinity itself looks flat and uninteresting. [...] it was just very very very big, so big that it gave the impression of infinity far better than infinity itself.
(Douglas Adams, *A Hitchhiker's Gudie to the Galaxy*)

Introduction

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About Me

I am Will.

Introduction

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Aim of This Talk

To help you to appreciate how big numbers get.

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I am Will.

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Rigour Level

Generally low.

The Rules

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Numbers

Numbers are natural numbers.

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Mathematics

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- A zillion.
- Infinity.
- The biggest number in this talk, plus one.

The Rules

Numbers
Numbers are natural numbers.

Mathematics
Numbers should be precisely specified.
"Precisely specified" is not precisely specified.

Bad "Numbers"
- A zillion.
- Infinity.
- The biggest number in this talk, plus one.
- The biggest number describable in ten English words, plus one.

How Big Do Numbers Get?

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As big as you like.

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Theorem

There is no biggest number.

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By contradiction. Suppose that N is the biggest number. Then $N+1$ is also a number, but $N+1 > N$. Contradiction. □

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Most Numbers Are Big

For a given number N, there are finitely many numbers smaller than N, and infinitely many numbers bigger than N.

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For a given number N, there are finitely many numbers smaller than N, and infinitely many numbers bigger than N.

So...

In the Big Number Game, you can never win. You can never even get close.

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For a given number N, there are finitely many numbers smaller than N, and infinitely many numbers bigger than N.

So...

In the Big Number Game, you can never win. You can never even get close. Let's get started.

The Beginning

The Beginning

- 0

The Beginning

- 0
- 1

The Beginning

- 0
- 1
- 2

The Beginning

- 0
- 1
- 2
- 3

The Beginning

- 0
- 1
- 2
- 3
- 4

The Beginning

- 0
- 1
- 2
- 3
- 4
- 5

The Beginning

- 0
- 1
- 2
- 3
- 4
- 5
- 6

The Beginning

- 0
- 1
- 2
- 3
- 4
- 5
- 6
- 7

The Beginning

- 0
- 1
- 2
- 3
- 4
- 5
- 6
- 7
- 8

The Beginning

- 0
- 1
- 2
- 3
- 4
- 5
- 6
- 7
- 8
- 9

The Beginning

- 0
- 1
- 2
- 3
- 4
- 5
- 6
- 7
- 8
- 9
- 10

The Beginning

- 0
- 1
- 2
- 3
- 4
- 5
- 6
- 7
- 8
- 9
- 10
- And so on.

Let's Go Faster

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How About...

646356798734978347943593245.

Let's Go Faster

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Bigger:

700000000000000000000000000.

Let's Go Faster

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646356798734978347943593245.

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700000000000000000000000000. I.e., $7 \cdot 10^{26}$.

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Bigger:

10^{27}.

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Lesson

Decimal notation isn't important. Exponentiation is.

Small Big Numbers

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- googol: 10^{100}
- googolplex: $10^{10^{100}}$

Exponentiation Example 1

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- $((10^8)^{(10^8)})^{(10^8)} = 10^{8 \cdot 10^{16}}$

Exponentiation Example 2

Definitions

$$\pi(x) := |\mathbb{P} \cap (-\infty, x]| \quad \mathrm{li}(x) := \int_0^x \frac{1}{\ln x}\, \mathrm{d}x$$

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Skewes's Number

For all small numbers $x \geq 2$, it appears that $\pi(x) \leq \mathrm{li}(x)$.
However, Littlewood proved that the sign of $\pi(x) - \mathrm{li}(x)$ changes infinitely often.

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However, Littlewood proved that the sign of $\pi(x) - \mathrm{li}(x)$ changes infinitely often.
An early upper bound for the smallest such x with $\pi(x) > \mathrm{li}(x)$ was given by Skewes:

$$e^{e^{e^{e^{7.705}}}} < 10^{10^{10^{964}}}.$$

Tetration

Recusive Definitions

$$\begin{aligned} n \cdot 0 &:= 0, & n \cdot (m+1) &:= n + (n \cdot m); \\ n^0 &:= 1, & n^{m+1} &:= n \cdot (n^m). \end{aligned}$$

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Let's Go Further

$${}^0n := 1, \qquad {}^{m+1}n := n^{{}^mn}.$$

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Let's Go Further

$${}^0 n := 1, \qquad {}^{m+1}n := n^{{}^m n}.$$

Example

$$10^{10^{100}} < 10^{10^{10^{10}}} = {}^4 10.$$

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Consider $^{3}3$.

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Consider ${}^{3}3$. Since

$${}^{3}3 = 3^{3^3} = 3^{27} = 7625597484987,$$

we have

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we have

$${}^{^3 3}3 = {}^{7625597484987}3.$$

This number is in the same "size class" as its logarithm.

Knuth Hyperoperators

Arrows

Exponentiation and tetration are denoted by arrows:

$$n \uparrow m = n^m \qquad n \uparrow\uparrow m = {}^m n.$$

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Pentation

Iterated tetration, pentation, can be written thusly:

$$3 \uparrow\uparrow\uparrow 4 = 3 \uparrow\uparrow (3 \uparrow\uparrow (3 \uparrow\uparrow 3)) = {}^{\text{number from last slide}}3$$

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Hexation

$$6 \uparrow\uparrow\uparrow\uparrow 9$$
$$= 6 \uparrow\uparrow\uparrow (6 \uparrow\uparrow\uparrow (6 \uparrow\uparrow\uparrow (6 \uparrow\uparrow\uparrow (6 \uparrow\uparrow\uparrow (6 \uparrow\uparrow\uparrow (6 \uparrow\uparrow\uparrow (6 \uparrow\uparrow\uparrow 6)))))))$$

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Comment

It's more common to start defining these at 1, but I found a way to start at 0.

Graham's Number

Setup

The *Graham–Rothschild Theorem* implies that there exists a number n such that any 2-colouring of the edges of the complete graph on the vertices of the n-dimensional hypercube contains a monochromatic coplanar K_4. Graham's number is an upper bound for the least such n.

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Definition

Define $g_0 := 4$, $g_{n+1} := 3 \uparrow^{g_n} 3$. Then *Graham's Number* is g_{64}.

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Lower bound

The best known lower bound for this problem is 13.

Conway Chained Arrows

Definition

$$() := 1 \qquad (a) := a \qquad (a \to b) := a^b$$
$$(X \to 1) := X \qquad (X \to 1 \to a) := X$$
$$(X \to (a+1) \to (b+1)) := (X \to (X \to a \to (b+1)) \to b)$$

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Relation to Knuth Arrows

$$(a \rightarrow b \rightarrow c) = a \uparrow^c b$$

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Hang on...

Is this notation even well-defined? Yes, but the easiest way to show this uses an idea we have yet to introduce.

Conway Chained Arrow Example 1

$$
\begin{aligned}
(3 \to 3 \to 1 \to 2) &= (3 \to 3) = 3^3 = 27 \\
(3 \to 3 \to (n+1) \to 2) &= (3 \to 3 \to (3 \to 3 \to n \to 2) \to 1) \\
&= 3 \uparrow^{(3 \to 3 \to n \to 2)} 3
\end{aligned}
$$

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\end{aligned}
$$

Comparison with Graham's Number

$$
(3 \to 3 \to 64 \to 2) < g_{64} < (3 \to 3 \to 65 \to 2)
$$

Conway Chained Arrow Example 2

$$
\begin{aligned}
(3 \to 3 \to 2 \to 3) &= (3 \to 3 \to (3 \to 3 \to 1 \to 3) \to 2) \\
&= (3 \to 3 \to 27 \to 2) \\
(3 \to 3 \to 3 \to 3) &= (3 \to 3 \to (3 \to 3 \to 2 \to 3) \to 2) \\
(3 \to 3 \to 2 \to 4) &= (3 \to 3 \to 27 \to 3) \\
(3 \to 3 \to 3 \to 4) &= (3 \to 3 \to (3 \to 3 \to 2 \to 4) \to 3)
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(3 \to 3 \to 3 \to 4) &= (3 \to 3 \to (3 \to 3 \to 2 \to 4) \to 3)
\end{aligned}
$$

$$(6 \to 6 \to 6 \to 6 \to 6 \to 6 \to 6 \to 6 \to 6) = (\text{a big number})$$

Other Notatations

- Bower's Eploding Array Notation (BEAF)

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Complaint

These notations are rather ad hoc. Is there a more systematic way to generate big numbers?

“The” Fast-Growing Hierarchy

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$$f_0(n) := n + 1;$$
$$f_{m+1}(n) := f_m^n(n).$$

In general, f_n is comparable with $n \mapsto n \uparrow^n n$.

Fast-Growing Hierarchy Examples

$$
\begin{aligned}
f_1(n) &= f_0^n(n) = 2n \\
f_2(n) &= f_1^n(n) = 2^n n > 2 \uparrow n \\
f_3(3) &= f_2(f_2(f_2(3))) = 2^{2^{2^3 3} 2^3 3} 2^{2^3 3} 2^3 3 \\
&> 2 \uparrow\uparrow 3 \\
f_{m+1}(n) &> 2 \uparrow^m n
\end{aligned}
$$

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f_{m+1}(n) &> 2 \uparrow^m n
\end{aligned}
$$

If FGH doesn't get us any faster speed than Knuth arrows, why bother? Because not all ordinals are finite.

Ordinals (Informal)

Comment

Despite only attempting to make large finite numbers, we now have occasion to introduce infinity, since it is useful.

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Counting

Numbers are used for counting. Ordinals are an extension of numbers which enable you to count past infinity.

$$0, 1, 2, 3, \ldots, \omega, \omega + 1, \ldots, \omega 2, \omega 2 + 1, \ldots, \omega 3, \omega 4, \ldots, \omega^2, \ldots$$

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$$0, 1, 2, 3, \dots, \omega, \omega + 1, \dots, \omega 2, \omega 2 + 1, \dots, \omega 3, \omega 4, \dots, \omega^2, \dots$$

Important Property

There are no infinite strictly-decreasing sequences of ordinals (i.e., any nonempty set of ordinals has a least element).

Fundamental Sequences

Successor and Limit Ordinals

An ordinal α is called a *successor ordinal* when there exists an ordinal β such that $\alpha = \beta + 1$. An ordinal is called a *limit ordinal* when it is not a successor ordinal.

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$$\begin{aligned}
\omega &: 0, 1, 2, \ldots \\
\omega 2 &: \omega, \omega + 1, \omega + 2, \ldots \\
\omega^2 &: \omega, \omega 2, \omega 3, \ldots \\
\omega^\omega &: 1, \omega, \omega^2, \omega^3, \ldots
\end{aligned}$$

FGH for Limit Ordinals

To the other two rules of the FGH, we add a third rule:

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Example

$$\begin{aligned} f_{\omega+1}(3) &= f_\omega(3) = f_\omega(f_\omega(f_3(3))) > f_\omega(f_\omega(2 \uparrow\uparrow 3)) \\ &= f_\omega(f_\omega(65536)) = f_\omega(f_{65536}(65536)) < f_\omega(2 \uparrow^{65535} 65536) \\ &< 2 \uparrow^{2\uparrow^{65535}65536-1} (2 \uparrow^{65535} 65536) \end{aligned}$$

Comparison with Previous Numbers

Graham's Number

The function $f_{\omega+1}$ is comparable with Graham's g. We have $f_{\omega+1}(64) > g_{64}$.

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The function $f_{\omega n}$ is compariable to Conway chained arrow notation with n arrows. The function f_{ω^2} *diagonalizes* over Conway chained arrow notation. Non-coincidentally, ω^2 is used in the most natural proof that the notation is well-defined.

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Transfinite Recursion on ω^2

For the arrow configuration $X \to a \to b$, assign the ordinal $\omega b + a$. Then the Conway notation is well-defined, since there are no infinite strictly decreasing sequences of ordinals.

Higher FGH Example

All previous notation got us merely up to ω^2. We have much greater power available to use now.

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Example

$$f_{\omega^\omega}(4)$$
$$= f_{\omega^4}(4) = f_{\omega^3 4}(4) = f_{\omega^3 3+\omega^2 4}(4) = f_{\omega^3 3+\omega^2 3+4\omega}(4) = f_{\omega^3 3+\omega^2 3+3\omega+4}(4)$$
$$= f_{\omega^3 3+\omega^2 3+3\omega+3}(f_{\omega^3 3+\omega^2 3+3\omega+3}(f_{\omega^3 3+\omega^2 3+3\omega+3}(f_{\omega^3 3+\omega^2 3+3\omega+3}(4))))$$

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Comment

This number is big.

Ordinals past ω^ω

To get large finite numbers, we now need large (countable) ordinals (and fundamental sequences for them).

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The sequence continues...

$$\ldots, \omega^\omega, \omega^\omega + 1, \ldots, \omega^\omega 2, \ldots, \omega^\omega 3, \ldots, \omega^{\omega+1}, \ldots, \omega^{\omega+1}2,$$

$$\ldots, \omega^{\omega+2}, \ldots, \omega^{\omega 2}, \ldots, \omega^{\omega^2}, \ldots, \omega^{\omega^2+1}, \ldots, \omega^{\omega^2 2}, \ldots, \omega^{\omega^3},$$

$$\ldots, \omega^{\omega^\omega}, \ldots, \omega^{\omega^{\omega^\omega}}, \ldots, \omega^{\omega^{\omega^{\omega^\omega}}}, \ldots$$

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The sequence continues...

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$$\ldots, \omega^{\omega+2}, \ldots, \omega^{\omega 2}, \ldots, \omega^{\omega^2}, \ldots, \omega^{\omega^2+1}, \ldots, \omega^{\omega^2 2}, \ldots, \omega^{\omega^3},$$
$$\ldots, \omega^{\omega^\omega}, \ldots, \omega^{\omega^{\omega^\omega}}, \ldots, \omega^{\omega^{\omega^{\omega^\omega}}}, \ldots$$

ε_0

The supremum of this sequence is called ε_0. This is the first fixed point of $\alpha \to \omega^\alpha$. It is associated with the fundamental sequence:

$$1, \omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ldots$$

Ordinal Representations

Ordinals less than ω^ω can be represented as finite sequences of numbers.

Ordinal Representations

Ordinals less than ω^ω can be represented as finite sequences of numbers.
Ordinals less than ε_0 can be represented as finite trees of numbers.

The Goodstein Function

Hereditary Base-n Notation

Write a number in base-n. Then write the exponents in base-n notation. Continue. Example:

$$69 = 2^6 + 2^2 + 2^1 = 2^{2^{2^1}+2^1} + 2^{2^1} + 2^1$$

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Next Step

Change the 2's to 3's, and subtract 1. So on:

$$3^{3^{3^1}+3^1} + 3^{3^1} + 3^1 - 1 = 3^{3^{3^1}+3^1} + 3^{3^1} + 2$$

$$4^{4^{4^1}+4^1} + 4^{4^1} + 2 - 1 = 4^{4^{4^1}+4^1} + 4^{4^1} + 1$$

$$= 5^{5^{5^1}+5^1} + 5^{5^1} + 1 - 1 = 5^{5^{5^1}+5^1} + 5^{5^1}$$

The Goodstein Function, Continued

Goodstein's Theorem

For any number n, the sequence contructed in the previous slide will always eventually reach 0.

The Goodstein Function $\mathscr{G}$

For any number n, $\mathscr{G}(n)$ is defined to be the number of steps required until the above sequence (starting with n) reaches 0.

Note

The function $\mathscr{G}$ has growth rate $\sim f_{\varepsilon_0}$. This can be understood as due to the "tree-like" structure of hereditary base-n notation.

Ordinals Past ε_0

Other Fixed Points

The function $\alpha \mapsto \omega^\alpha$ is a *normal function*, so it has arbitrarily many fixed points (see MATH3306). The ordinal ε_0 is the first such fixed point. The next is ε_1. A typical fundamental sequence for this is

$$\varepsilon_0 + 1, \omega^{\varepsilon_0+1}, \omega^{\omega^{\varepsilon_0+1}}, \ldots$$

Note that $w^{\varepsilon_0} = \varepsilon_0$. Fundamental sequences for other "ε-numbers" can be constructed similarly.

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ε fixed points

The function $\alpha \to \varepsilon_\alpha$ is also normal, so it to has arbitrarily many fixed points. The first such fixed point is denoted by ζ_0, with fundamental sequence

$$0, \varepsilon_0, \varepsilon_{\varepsilon_0}, \varepsilon_{\varepsilon_{\varepsilon_0}}, \dots,$$

Of course, we can then define ζ_1. And $\zeta_{\zeta_{\zeta,\dots}}$.

The Veblen Hierarchy

We have the ε ordinals, the ζ ordinals, etc. This can be seen as the start of *another* infinite hierarchy.

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This is the first ordinal α such that $\varphi_\alpha(0) = \alpha$. In other words, it enables the Veblen function to "eat itself". It is denoted Γ_0.

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Higher Countable Ordinals

One can define higher countable ordinals, with correspondingly more complicated notations and fundamental sequences.

Proof Theoretic Ordinals

Warning

This content will be more vague than usual. It may not be accurate.

Inserted After Talk...

(In fact, the following slides were not accurate, so I've taken the liberty of removing them.)

Conclusion?

Numbers have no end. But this talk must.

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How big *do* numbers get?

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Very…

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