Hyperplane Arrangements

A Link Between Combinatorics and Topology

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A Problem in the Plane

Suppose we have m randomly drawn lines in $\mathbb{R}^2$. With probability 1, no two lines will be parallel and no three will meet at point.

How many regions does this cut the plane into?

$m = 3$ Example

We can count that $r(\mathcal{A}) = 7$, what about for a general m?

General m

$$r(\mathcal{A}) = \underbrace{1}_{\text{plane}} + \underbrace{1}_{\text{1st line}} + \underbrace{2}_{\text{2nd line}} + \underbrace{???}_{\text{other lines}}$$

j^{th} Line

1. Start drawing the j^{th} line far away from all others.
2. Any time we intersect a line, we split the region we were just in into two, adding a region.
3. This happens $j-1$ times for each of the lines already drawn.
4. Then we split the region that we travel to infinity in into two, adding one region.

In total the j^{th} line adds $(j-1)+1=j$ regions!

General m

$$
\begin{aligned}
r(\mathcal{A}) &= \underbrace{1}_{\text{plane}} + \underbrace{1}_{\text{1}^{\text{st}}\text{ line}} + \underbrace{2}_{\text{2}^{\text{nd}}\text{ line}} + \cdots + \underbrace{j}_{j^{\text{th}}\text{ line}} + \cdots + \underbrace{m}_{m^{\text{th}}\text{ line}} \\
&= 1 + \sum_{j=1}^{m} j \\
&= 1 + \frac{m(m+1)}{2}
\end{aligned}
$$

A hyperplane is an $(n-1)$-dimensional affine subspace of $\mathbb{R}^n$ (a translational of an $(n-1)$-dimensional vector subspace). More concretely, let $v_1, \ldots, v_n, a \in \mathbb{R}$ with not all v_i zero. Then the set of points $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ satisfying

$$v_1 x_1 + \cdots + v_n x_n = a$$

is a hyperplane. (Think a line in $\mathbb{R}^2$.)

A hyperplane arrangement $\mathcal{A}$ is a finite set of hyperplanes. (Think the set of m lines in $\mathbb{R}^2$.)

Generalising to $\mathbb{R}^n$

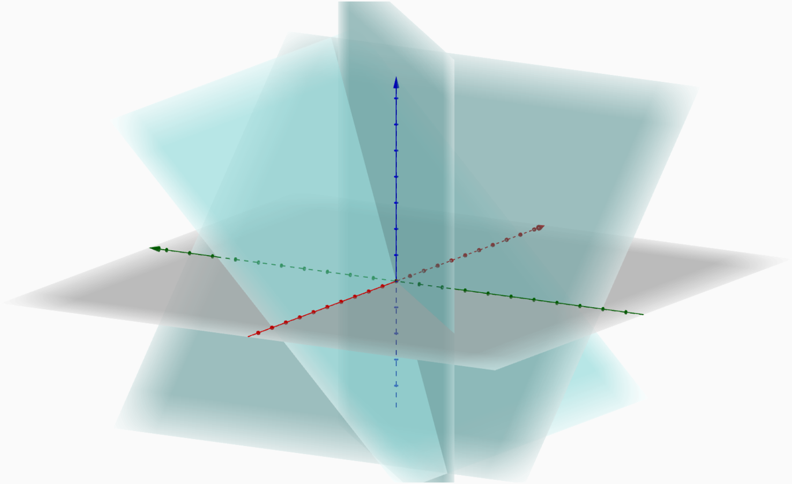
Let $\mathcal{A} = \{H_1, \ldots, H_m\}$ be an arrangement. We can define the hyperplane complement M,

$$M = \mathbb{R}^n \setminus \bigcup_{H_i \in \mathcal{A}} H_i$$

(Think all the empty space in $\mathbb{R}^2$ after drawing the lines.)

A region is a connected component of the hyperplane complement M. (Think one of the pieces that the lines cut the plane into.)

An Example in $\mathbb{R}^3$

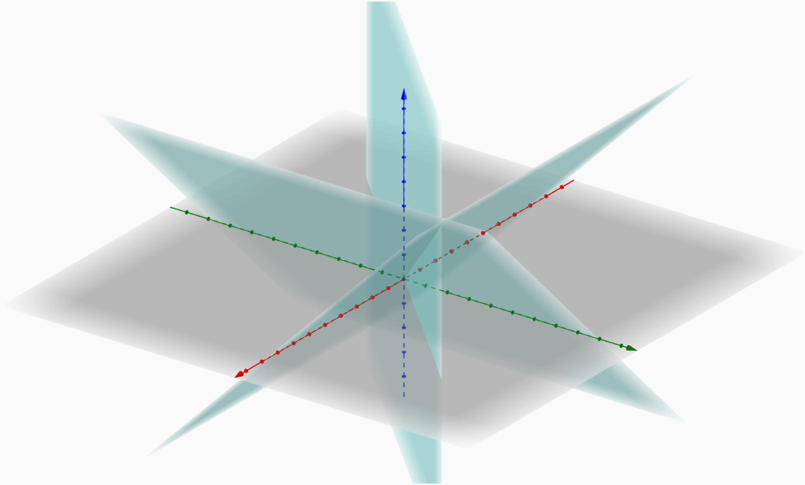


Hyperplane equations:

$$x_1 = x_2, \quad x_2 = x_3, \quad x_1 = x_3$$

An Example in $\mathbb{R}^3$

A different perspective...



We can count that $r(\mathcal{A}) = 6$.

A Harder Example...

Define the braid arrangement in $\mathbb{R}^n$ to be

$$\mathcal{B}_n = \{x_i = x_j \,|\, 1 \leq i < j \leq n\}$$

The example we just saw is $\mathcal{B}_3$.

How do we count the number of regions in $\mathcal{B}_n$?

An Observation About Regions

A hyperplane divides $\mathbb{R}^n$ into two half-spaces.

For the plane $x_i = x_j$, the corresponding half-spaces would be

$$x_i > x_j, \quad x_i < x_j$$

The key observation is that $x, y \in \mathbb{R}^n$ are in the same region if and only if for each hyperplane, x and y are in the same half-space.

Counting Regions for $\mathcal{B}_n$

Let $k = (k_1, \ldots, k_n)$ be in the hyperplane complement of $\mathcal{B}_n$. Since $\mathcal{B}_n$ is all $x_i = x_j$ hyperplanes, all k_i are distinct!

The components can be ordered

$$k_{i_1} < k_{i_2} < \cdots < k_{i_n},$$

for some order of indices $\{i_1, \ldots, i_n\} = \{1, \ldots, n\}$.

Any $l = (l_1, \ldots, l_n)$ satisfying the same ordering

$$l_{i_1} < l_{i_2} < \cdots < l_{i_n}$$

is in the same region as k.

Why? Because k and l satisfy the same $x_i > x_j$ or $x_i < x_j$ relations!

Counting Regions for $\mathcal{B}_n$

If $m = (m_1, \ldots, m_n)$ has a different ordering of components

$$m_{j_1} < m_{j_2} < \cdots < m_{j_n}$$

then there is a least one i, j pair such that, say, $k_i > k_j$ while $m_i < m_j$.

So k and m are in different regions!

The upshot — the number of regions is the same as number of orderings of indices, so $r(\mathcal{B}_n) = n!$.

The Intersection Poset

For an arrangement $\mathcal{A}$, the intersection poset $L(\mathcal{A})$ is the set

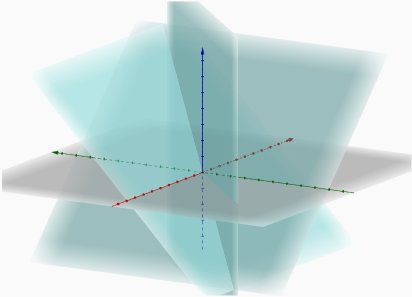
$$L(\mathcal{A}) = \{\mathbb{R}^n\} \cup \{H_{i_1} \cap \cdots \cap H_{i_s} \neq \emptyset \mid H_{i_k} \in \mathcal{A}, s \geq 1\}$$

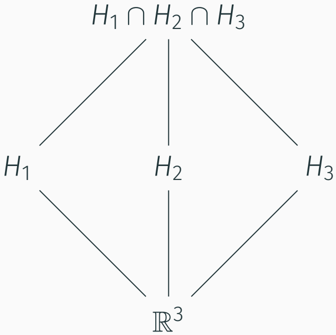
equipped with the partial order of reverse inclusion. For $x, y \in L(\mathcal{A})$, $x \leq y$ if $x \supseteq y$.

This is a combinatorial object, not geometric/topological.

$\mathbb{R}^n$ is the minimal element since for any $x \in L(\mathcal{A})$, $\mathbb{R}^n \supseteq x$, which means $\mathbb{R}^n \leq x$.

$L(\mathcal{B}_3)$





The Möbius Function

For $x, y \in L(\mathcal{A})$ with $x \leq y$, define the Möbius function by

$$\mu(x,y) = \begin{cases} 1 & \text{if } x = y, \\ -\displaystyle\sum_{x \leq t < y} \mu(x,t) & \text{otherwise.} \end{cases}$$

The key characteristic of this function is the Möbius inversion formula. If $f, g : L(\mathcal{A}) \to \mathbb{R}$, then

$$f(x) = \sum_{t \geq x} g(t)$$

is equivalent to

$$g(x) = \sum_{t \geq x} \mu(x,t) f(t)$$

The Characteristic Polynomial

For an arrangement $\mathcal{A}$, define the characteristic polynomial $p_{\mathcal{A}}(t)$ as

$$p_{\mathcal{A}}(t) = \sum_{x \in L(\mathcal{A})} \mu(\mathbb{R}^n, x)\, t^{\dim(x)}$$

This polynomial encodes information about the Möbius function in the coefficients.

The Big Theorem

In 1975, Thomas Zaslavsky proved the following theorem:

Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$. Then, the number of regions is given by

$$r(\mathcal{A}) = (-1)^n p_{\mathcal{A}}(-1)$$

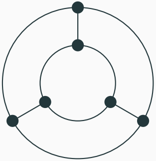
This is our dictionary to translate between topology and combinatorics.

A Topological Tool

We're going to need the Euler characteristic, denoted χ. For a shape in the plane, decompose into vertices, edges and faces. Then χ is defined by

$$\chi = \#V - \#E + \#F$$

For example, we can decompose an annulus as so:



The Euler characteristic is

$$\chi = \#V - \#E + \#F = 6 - 9 + 3 = 0$$

Euler Characteristic in $\mathbb{R}^n$

For a topological space, fix a decomposition. Let k_i be the number of i-dimensional "cells". Then χ is defined as

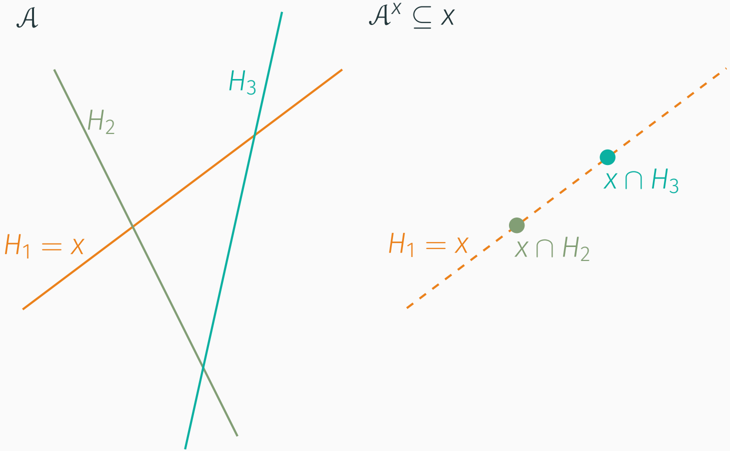
$$\chi = k_0 - k_1 + k_2 - \cdots = \sum_i (-1)^i k_i$$

For nice spaces and decompositions, this is independent of the exact choice of decomposition!

The Subarrangement $\mathcal{A}^x$

For any $x \in L(\mathcal{A})$, define the subarrangement $\mathcal{A}^x$ by

$$\mathcal{A}^x = \{x \cap H \mid H \in \mathcal{A} \text{ and } x \cap H \neq \emptyset, x \cap H \neq x\}$$



Proving the Theorem

By decomposing x as a single $\dim(x)$-dimensional cell, we have

$$\chi(x) = \sum_i (-1)^i k_i = (-1)^{\dim(x)}$$

$\mathcal{A}^x$ is a decomposition that we can use to compute $\chi(x)$.

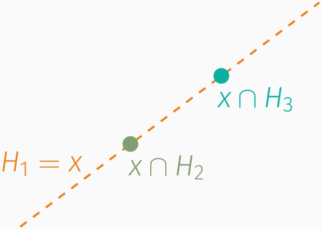
$r(\mathcal{A}^x)$ is the number of $\dim(x)$-dimensional cells that make up x.

To count all cells of all dimensions, sum over $t \geq x$ (cells $t \subseteq x$).

So we also have

$$\chi(x) = \sum_i (-1)^i k_i = \sum_{t \geq x} (-1)^{\dim(t)} r(\mathcal{A}^t)$$

Proving the Theorem



$$\chi(X) = \sum_{t \geq X} (-1)^{\dim(t)} r(\mathcal{A}^t)$$
$$= (-1)^1 \cdot 3 + (-1)^0 \cdot 1 + (-1)^0 \cdot 1$$
$$= -1$$

Matches $\chi(X) = (-1)^{\dim(X)} = (-1)^1$

For any $x \in L(\mathcal{A})$, by computing $\chi(x)$ in two ways we have that

$$(-1)^{\dim(x)} = \sum_{t \geq x} (-1)^{\dim(t)} r(\mathcal{A}^t)$$

By Möbius inversion we get

$$(-1)^{\dim(x)} r(\mathcal{A}^x) = \sum_{t \geq x} \mu(x, t) \, (-1)^{\dim(t)}$$

Proving the Theorem

We want to know $r(\mathcal{A}) = r(\mathcal{A}^{\mathbb{R}^n})$, so set $x = \mathbb{R}^n$ to get

$$(-1)^{\dim(\mathbb{R}^n)} r(\mathcal{A}) = \sum_{t \geq \mathbb{R}^n} \mu(\mathbb{R}^n, t)(-1)^{\dim(t)}$$

But $\dim(\mathbb{R}^n) = n$ and $t \geq \mathbb{R}^n$ is all $t \in L(\mathcal{A})$ since $\mathbb{R}^n$ is the minimal element. Substituting and rearranging gives

$$r(\mathcal{A}) = (-1)^n \sum_{t \in L(\mathcal{A})} \mu(\mathbb{R}^n, t)(-1)^{\dim(t)},$$

which by definition means

$$r(\mathcal{A}) = (-1)^n p_{\mathcal{A}}(-1),$$

proving the theorem!

References

- *An Introduction to Hyperplane Arrangements*, Richard P. Stanley, https://math.mit.edu/~rstan/arrangements/arr.html, 2006.
- *Arrangements of Hyperplanes*, Peter Orlik, Hiroaki Terao, Springer-Verlag, 1992.