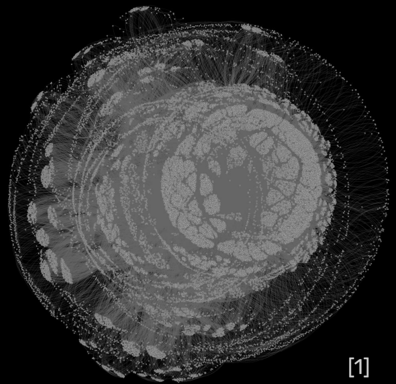
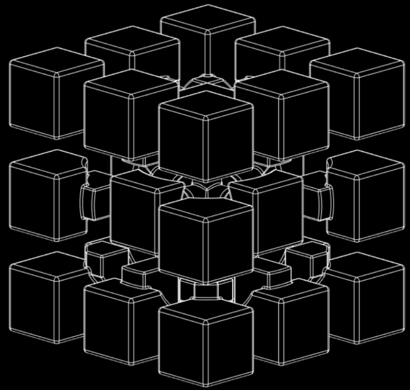
exploring rubik's cube with group theory

adriel efendy & benjamin paul



[1]

fun fact while we wait: the maximum number of moves required to solve the cube from any state, equal to the diameter of the cube group's Cayley graph, is **20**.



[2]

about us

bpaul 🤯

cuber since 9 years ago

math / cs student since 12 weeks ago

adriel/peko 🦆

cuber since 5 years ago

metr. engg. / cs student since 3 years ago

math2301 sufficient, ask questions as we go!

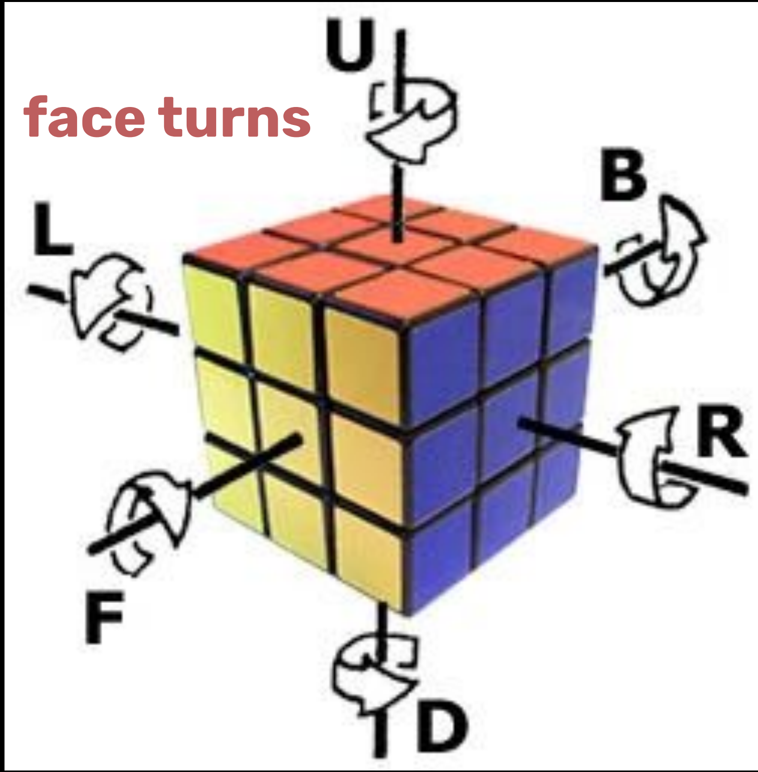
what’s a rubik’s cube?

puzzle popular throughout 80s

invented by Ernő Rubik

first “scramble” by turning random sides,
then “solve” by restoring solid colours on every face

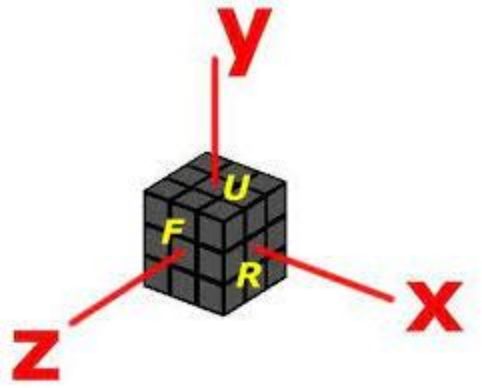
move notation



https://rubiks.fandom.com/wiki/Notation?file=Rubik%27s_cube_notation.jpg

slice moves

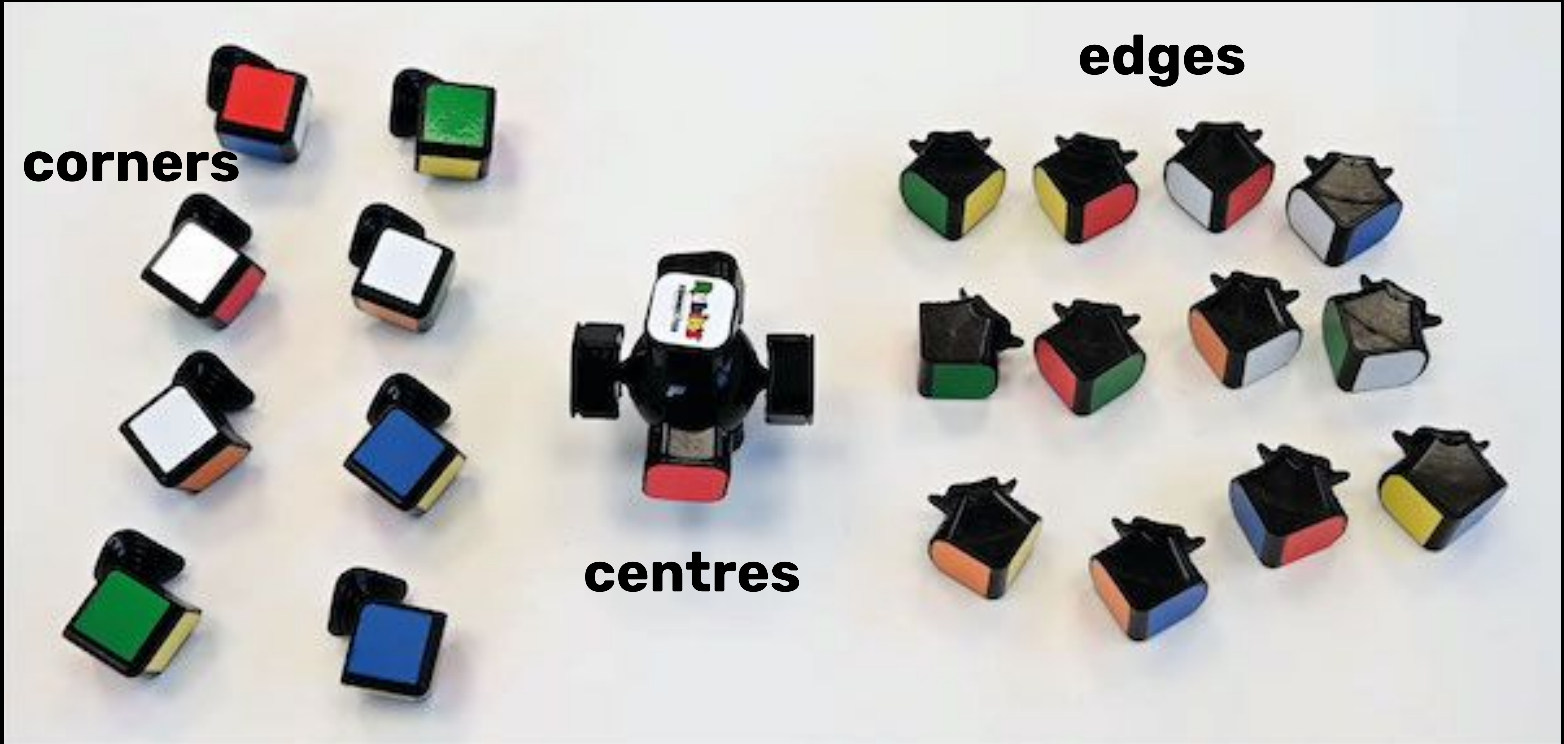
https://chroniclesofcalculation.files.wordpress.com/2015/09/notation.png



cube rotations

https://sites.google.com/site/thethreelayers/Home/notation-guide/3x3x3/rotation.JPG

pieces



www.digiadzo.com/how-to-take-apart-a-rubik%2527s-cube-k.html

composition of moves

consider the moves generated by

$$\{F, B, U, D, L, R\}$$

✅ closure (trivial)

composition of moves

consider the moves generated by

$$\{F, B, U, D, L, R\}$$

✅ closure (trivial)

✅ inverses A A′ = A′ A = I

✅ associativity A (B C) = (A B) C

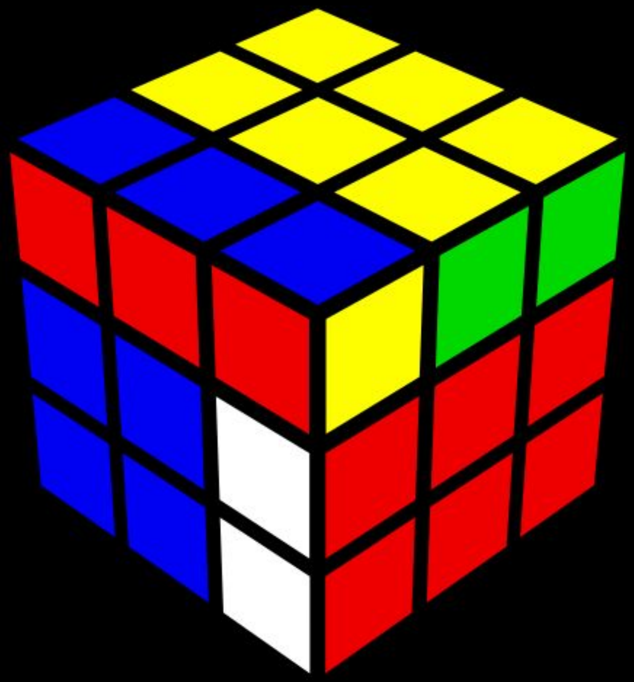
this forms a group! 🤯

(with composition as the group operation)

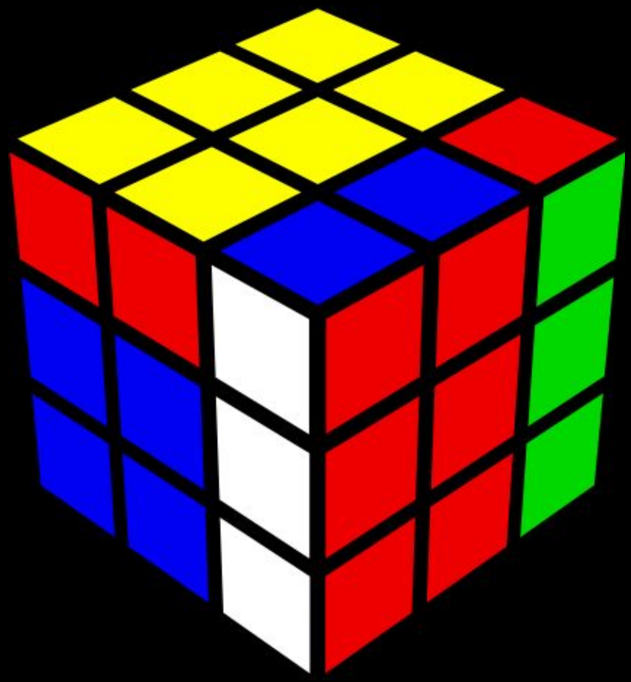
(this group *acts* on the cube)

(R U) (U′ R′)

❌ commutativity



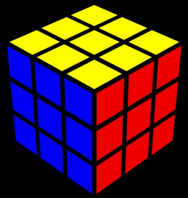
R U



U R

no centres!

one approach to formalise

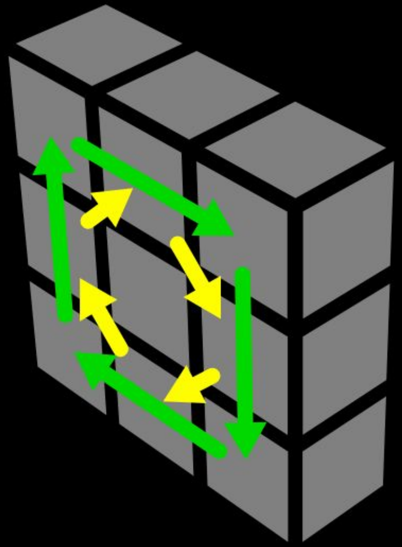


fix an orientation of the cube.

there are $6 \times 8 = 48$ *non-centre* stickers on the cube.

define $\mathcal{G} = \langle \{F, B, U, D, L, R\} \rangle$, where

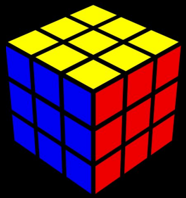
$$F = (\mathrm{UF\ RF\ DF\ LF})(\mathrm{FU\ FR\ FD\ FL})$$
$$(\mathrm{UFL\ RFU\ DFR\ LFD})$$
$$(\mathrm{UFR\ RFD\ DFL\ LFU})$$
$$(\mathrm{FLU\ FRU\ FRD\ FRL})$$



etc., expressing each as four-cycles.

$$\mathcal{G} \leq S_{48}$$

one approach to formalise



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$$
\begin{aligned}
F = &(\mathrm{UF\ RF\ DF\ LF})(\mathrm{FU\ FR\ FD\ FL}) \\
&(\mathrm{UFL\ RFU\ DFR\ LFD}) \\
&(\mathrm{UFR\ RFD\ DFL\ LFU}) \\
&(\mathrm{FLU\ FRU\ FRD\ FRL})
\end{aligned}
$$

simple to implement but difficult to reason about. we want more structure.

etc., expressing each as four-cycles.

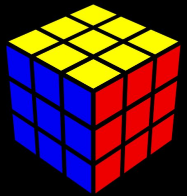
$$\mathcal{G} \leq S_{48}$$

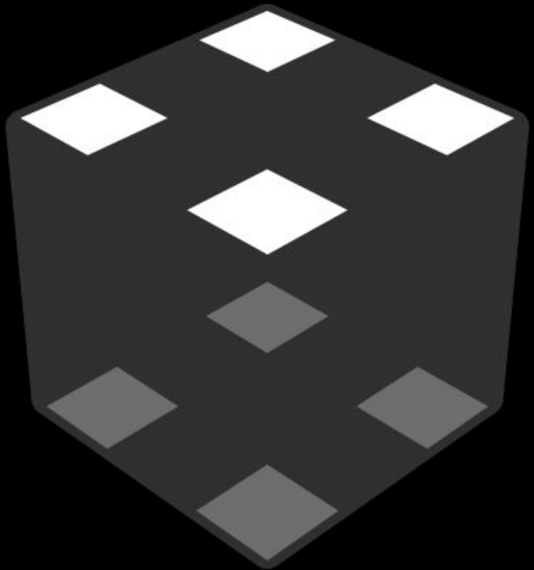
orientations

consider corner **orientations**, again fixing a cube orientation.

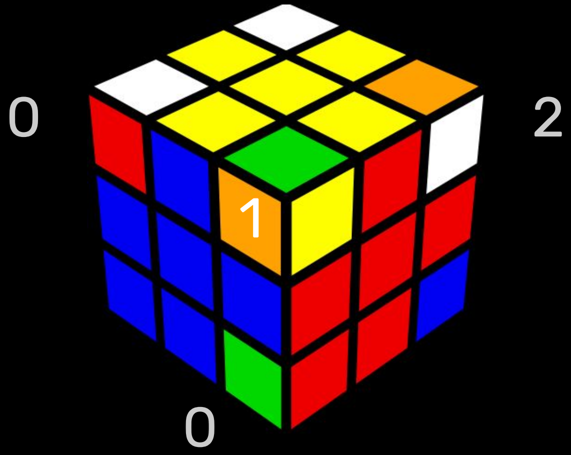
want to systematically enumerate these orientations.

orientations





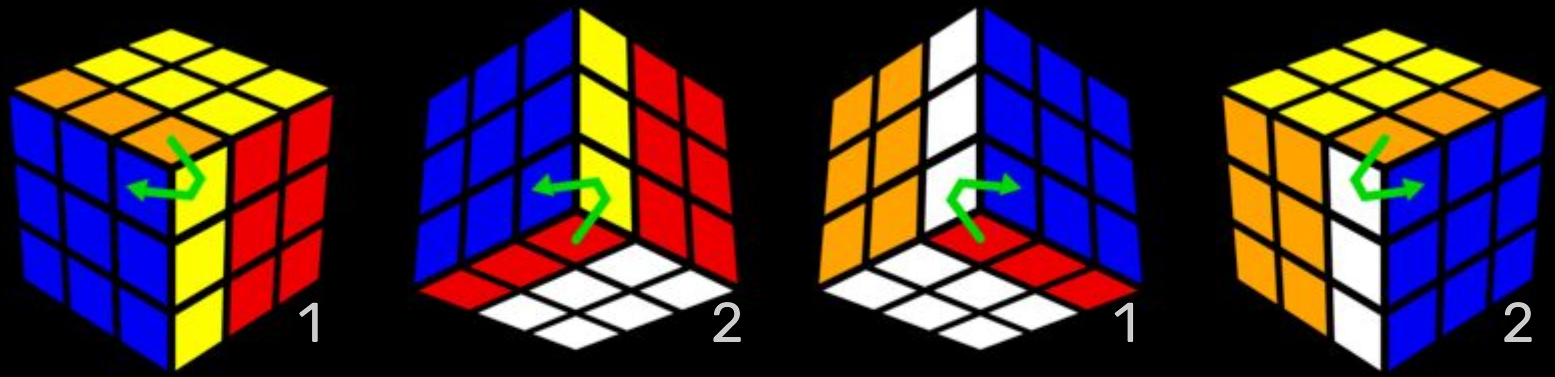
orientation 0 if its U or D coloured sticker is at the highlighted positions.



eight corners, so we describe it as $\mathbb{Z}_3^8$, right?

orientations

a single face turn modifies orientation:

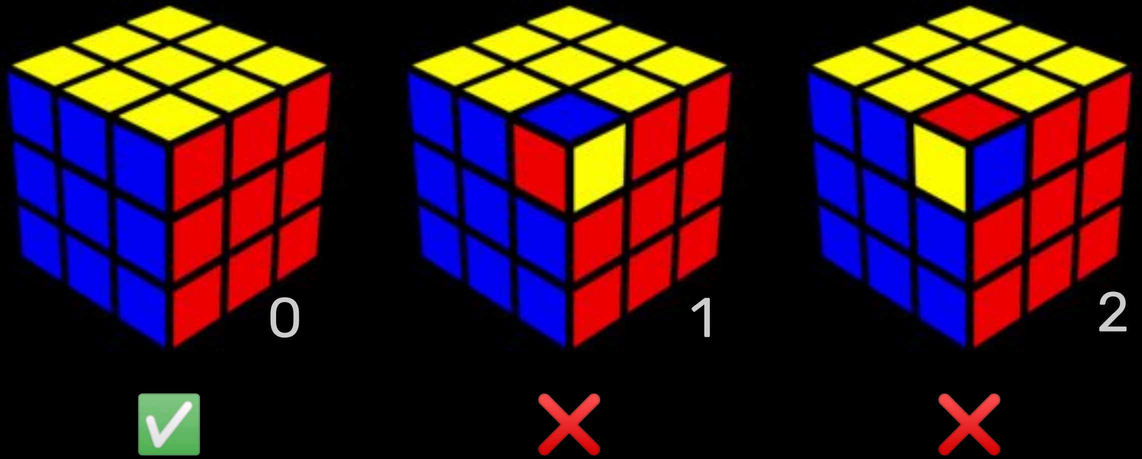


$$1 + 2 + 1 + 2 = 6 \equiv 0 \pmod 3$$

sum of CO, modulo 3, is ***invariant*** under face turns.

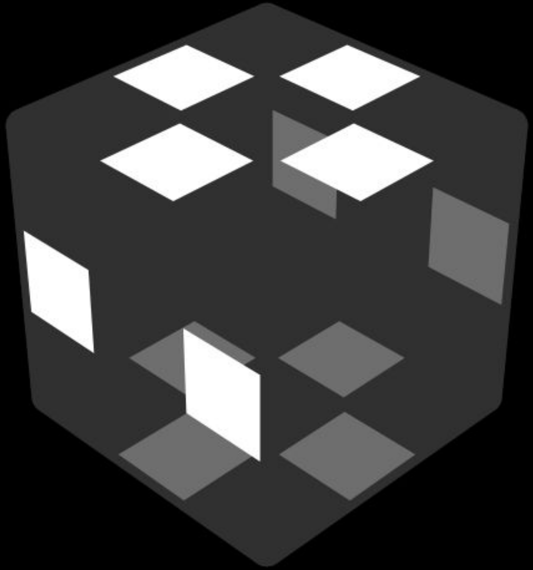
orientations

three equivalence classes:



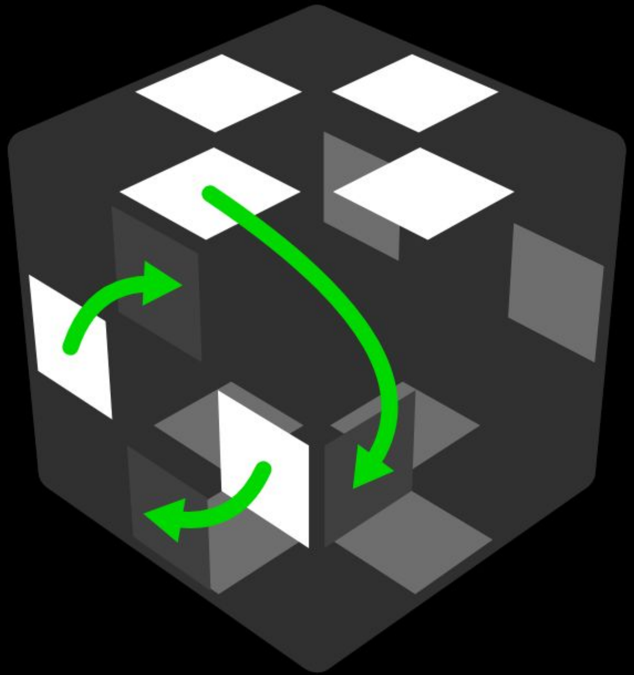
describe corner orientation as $\mathbb{Z}_3^8/\mathbb{Z}_3 \cong \mathbb{Z}_3^7$

orientations



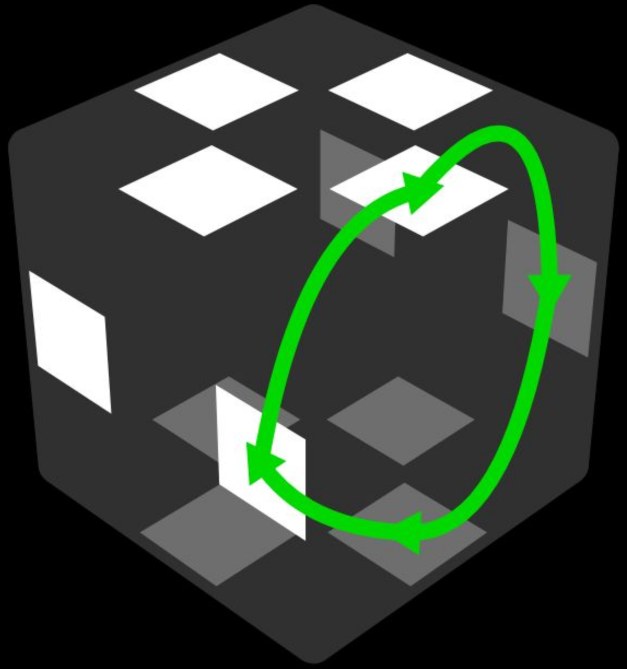
0 if U/D colour at highlighted.
if edge from E slice, look at F/B colour.

orientations



F/B: flip 2 edges 0→1,
2 edges 1→0

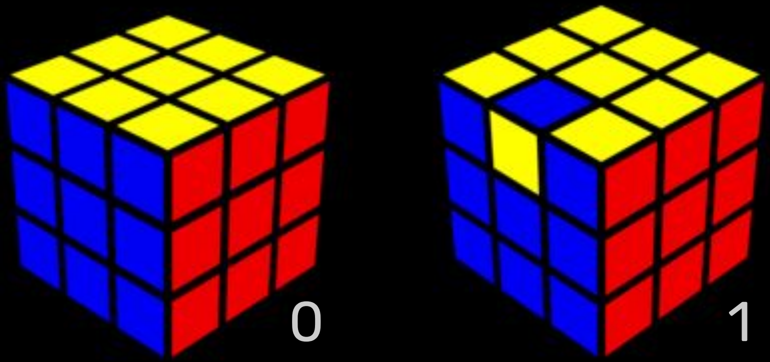
EO sum mod 2
invariant



else: flip 0 edges

orientations

two equivalence classes:

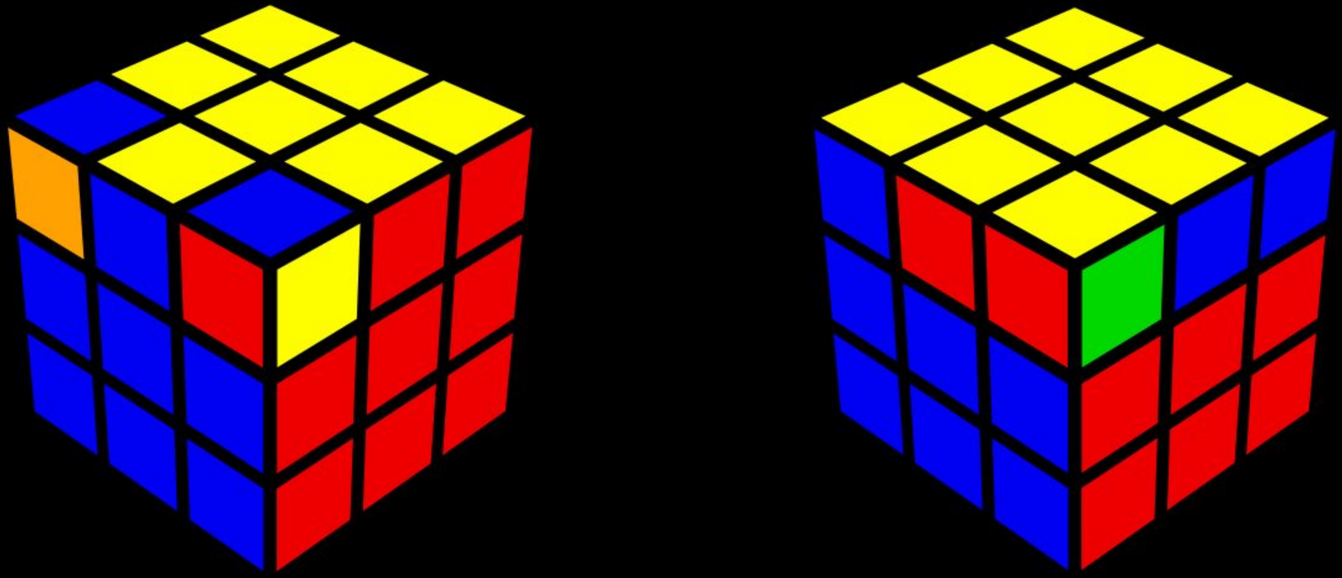


✅ ❌

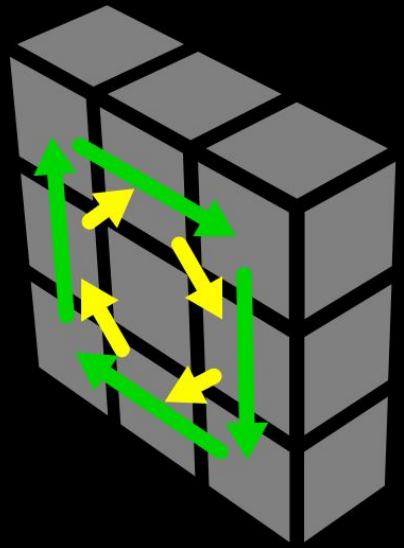
$$\mathbb{Z}_2^{12}/\mathbb{Z}_2 \cong \mathbb{Z}_2^{11}$$

permutations

permutation: movements of pieces, not considering orientation. also means an element of S_n.



permutations



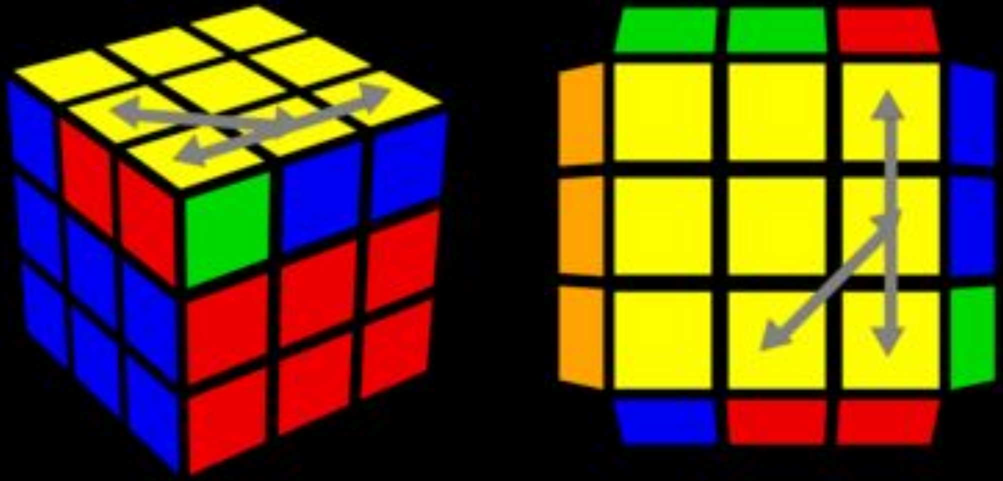
4-cycle in edges (3 transpositions)
4-cycle in corners (3 transpositions)

parities of the edge and corner permutations must be equal!

$(\operatorname{sgn}(\sigma_e) + \operatorname{sgn}(\sigma_c)) \bmod 2$ is invariant under face turns

permutations

this sequence flips both edge and corner parities:



R U R' F' R U R' U' R' F R2 U' R' U'

J perm, “cycle structure” 2e2c

permutations

describe subgroup of (cube) permutations by:

even corner perms

$(A_8$

permutations

describe subgroup of (cube) permutations by:

even corner perms — even edge perms

$$(A_8 \times A_{12})$$

permutations

describe subgroup of (cube) permutations by:

$$(A_8 \times A_{12}) \rtimes \mathbb{Z}_2$$

- A_8: even corner perms
- A_{12}: even edge perms
- $\rtimes$: outer semidirect product
- $\mathbb{Z}_2$: 2e2c swap to change *both* parities

for groups H and K, $H \rtimes K$ is the direct product $H \times K$
but with a new group operation involving conjugating $k \in K$.

(ask us later if interested)

piece perspective

CO EO even CP even EP 2e2c

$$\mathcal{G} \cong \left(\mathbb{Z}_3^7 \times \mathbb{Z}_2^{11}\right) \rtimes \left(\left(A_8 \times A_{12}\right) \rtimes \mathbb{Z}_2\right)$$

$$|\mathcal{G}| = 3^7 \cdot 2^{11} \cdot \frac{8!}{2} \cdot \frac{12!}{2} \cdot 2 \approx 4.33 \cdot 10^{19}$$

$$\mathcal{G}^* \cong \mathbb{Z}_3^8 \times \mathbb{Z}_2^{12} \times S_8 \times S_{12}$$

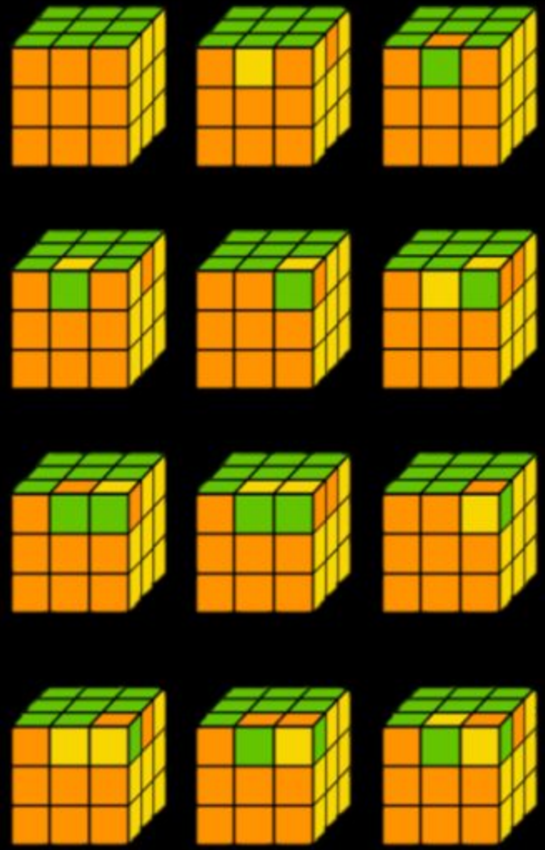
$$|\mathcal{G}^*| = 3^8 \cdot 2^{12} \cdot 8! \cdot 12! = 12|\mathcal{G}|$$

solvability

$$|\mathcal{G}^*| = 12|\mathcal{G}|$$

1 in 12 states are solvable

check CO, EO, perm. parity to determine solvability



representatives of cosets of $\mathcal{G}$ in $\mathcal{G}^*$

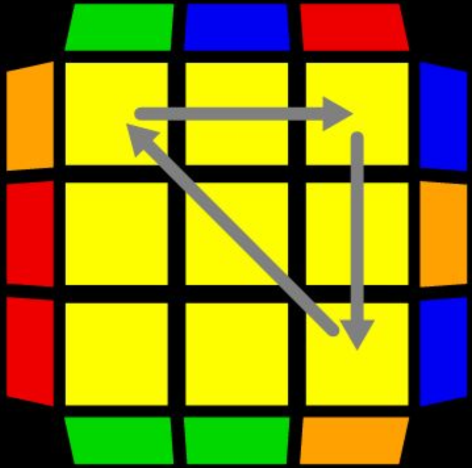
conjugates

the *conjugate* of X and Y is [X: Y] = X Y X’.

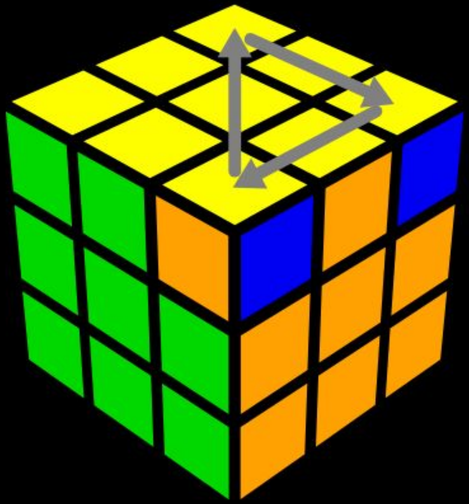
X “sets up” the cube to change the pieces that Y cycles.

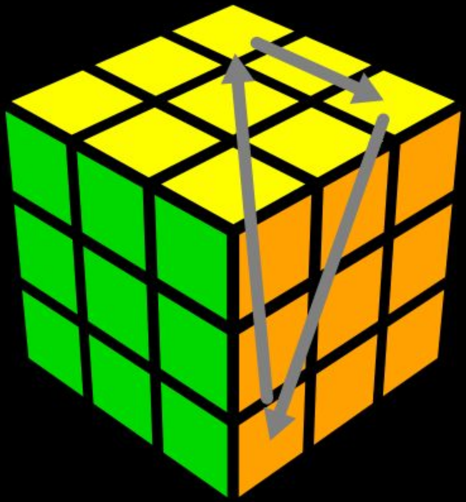
conjugates

let Y be the sequence R′ F R′ B2 R F′ R′ B2 R2.



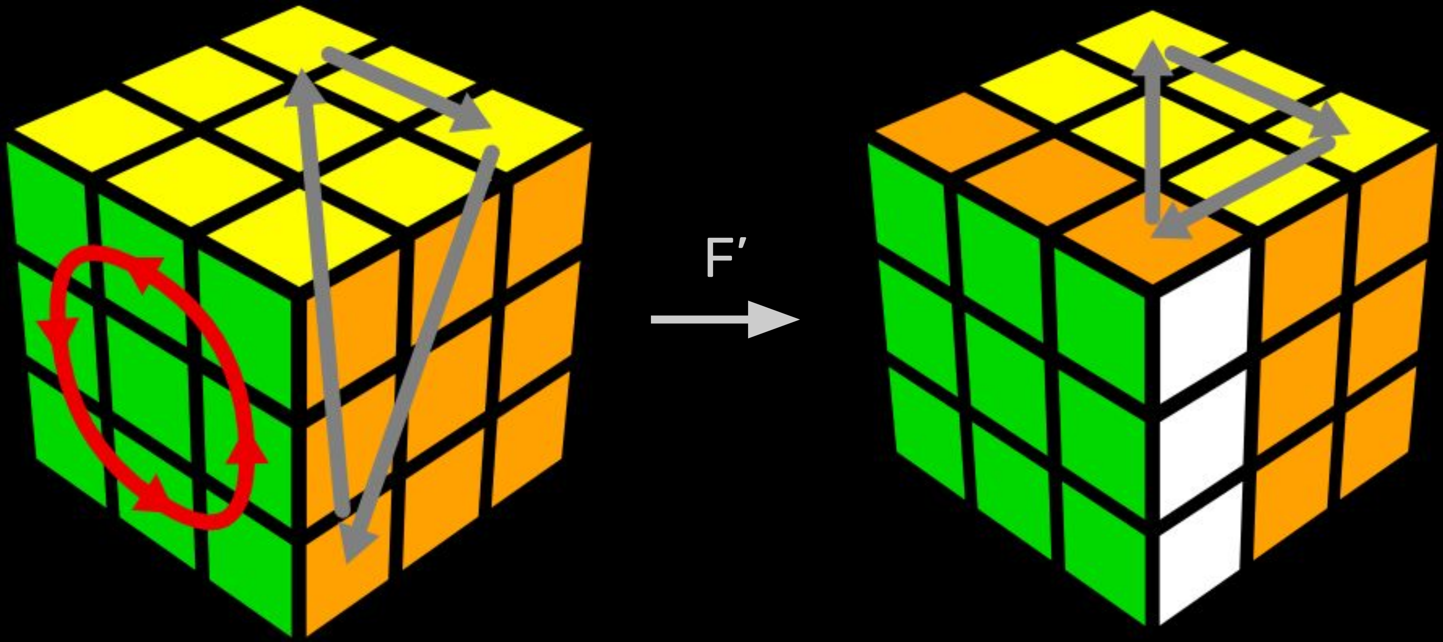
viewed from U face



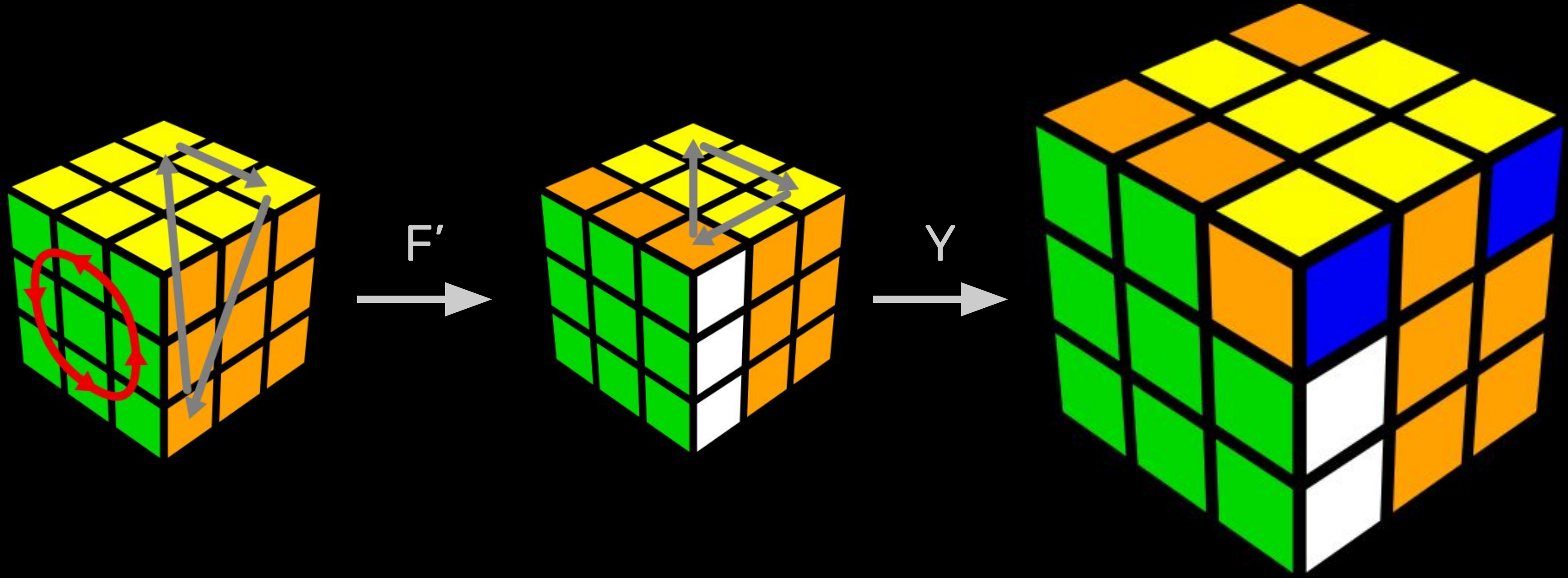


how to make this?

conjugates

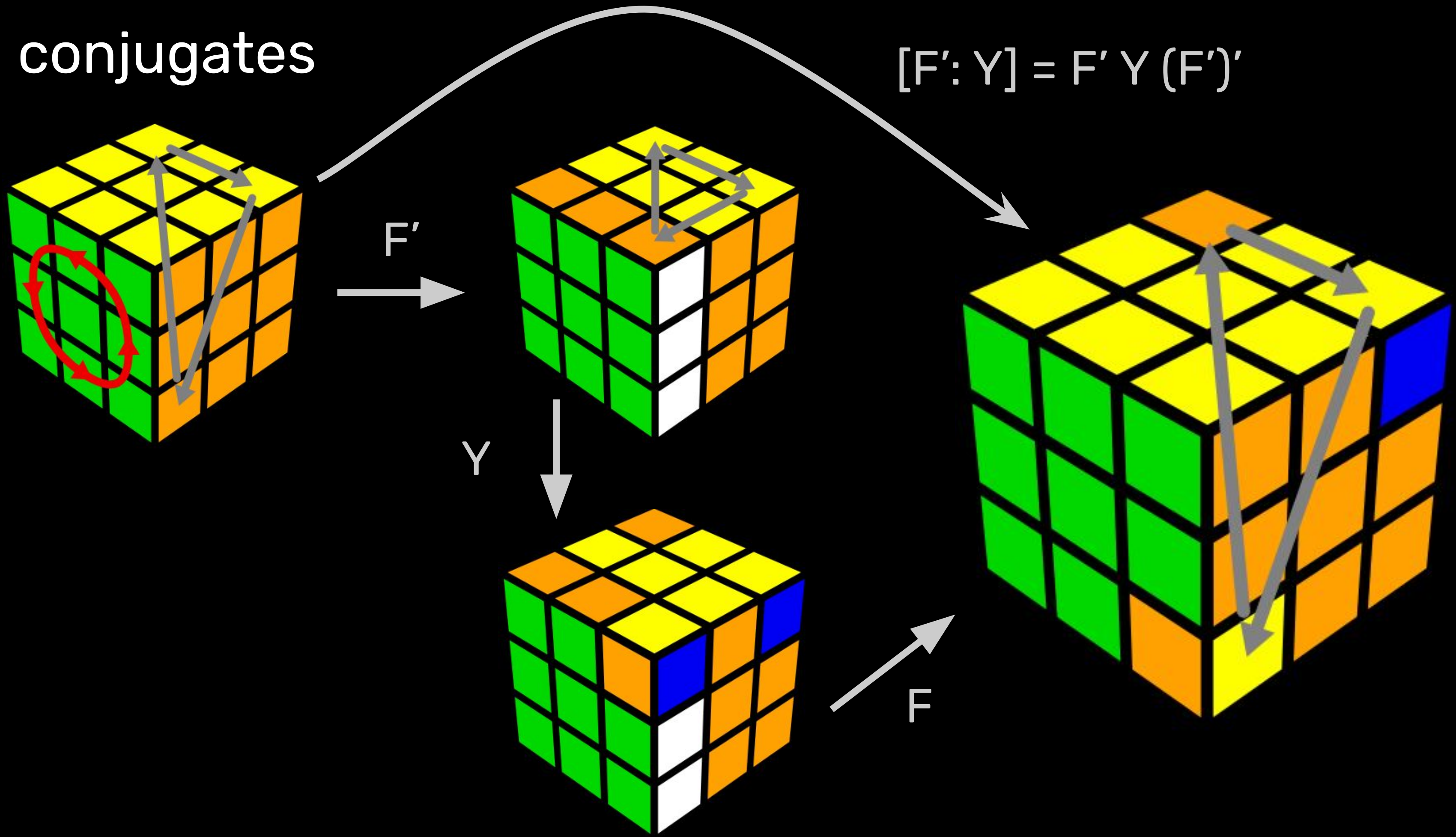


conjugates



need to restore initial setup with (F′)′ = F

conjugates



conjugates

thm: $\alpha, \beta \in S_n$ are conjugates ($\exists\, \gamma \in S_n : \alpha = [\gamma\colon \beta]$)
iff they have the same cycle structure.

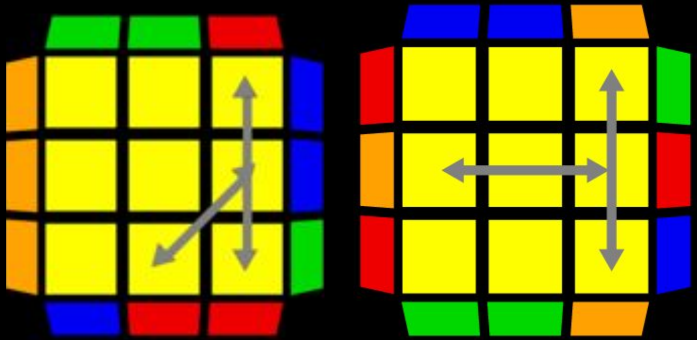
$$\alpha = (1\,2\,3)(4\,5)(6\,7)$$
$$\beta = (1\,3\,6)(2\,4)(5\,7)$$

cycle structure (3, 2, 2)

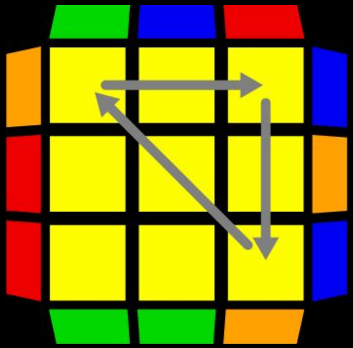
$$\gamma = (2\,3\,6)(4\,5\,7)$$
$$\alpha = \gamma\beta\gamma^{-1}$$

conjugates

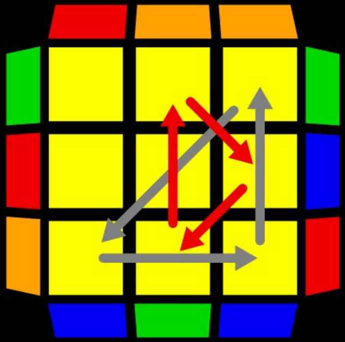
cube "cycle structure" considers piece permutation *and orientation*



2e2c



0e3c



3e3c

precise definition is beyond scope, but a similar result holds.

conjugation isn't enough!

commutators

the *commutator* of moves X and Y is [X, Y] = X Y X′ Y′

let $\mathrm{fix}(X)$ be the fixed points of X, from piece or sticker perspective.
let $\mathrm{mov}(X)$ be the complement of $\mathrm{fix}(X)$.
defining $\mathrm{mov}(X, Y) = \mathrm{mov}(X) \cap \mathrm{mov}(Y)$, we have

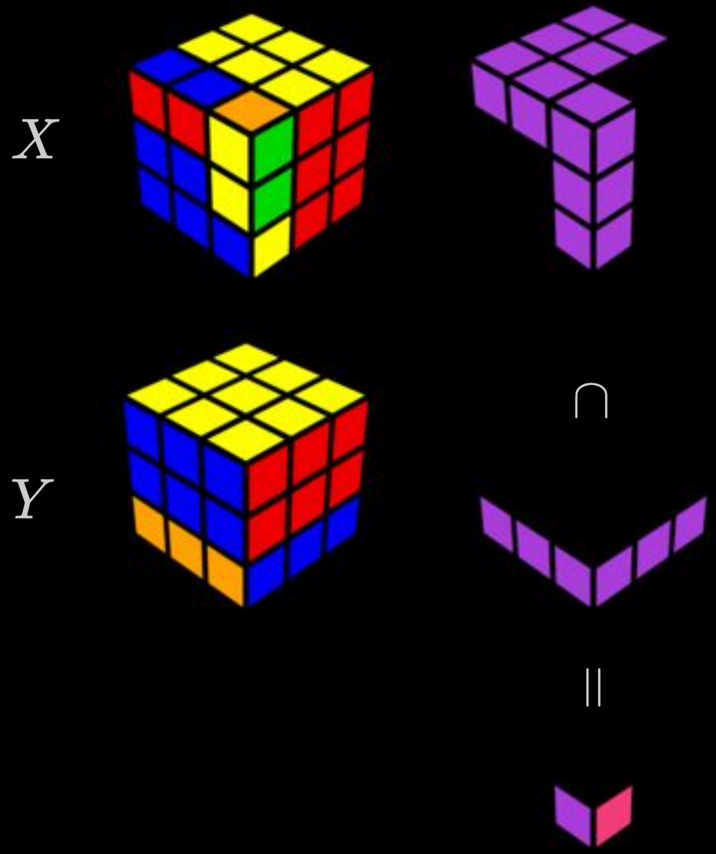
pick X and Y to control this

$$\mathrm{mov}([X, Y]) \subseteq \underbrace{\boxed{\mathrm{mov}(X, Y)}}_{\text{pieces moved by both}} \cup \underbrace{X' \,\mathrm{mov}(X, Y) \cup Y' \,\mathrm{mov}(X, Y)}_{\text{pieces that X and Y take to those positions}}$$

$$\operatorname{mov}([X,Y]) \subseteq \operatorname{mov}(X,Y) \cup X' \operatorname{mov}(X,Y) \cup Y' \operatorname{mov}(X,Y)$$

commutators

[R U R′, D]



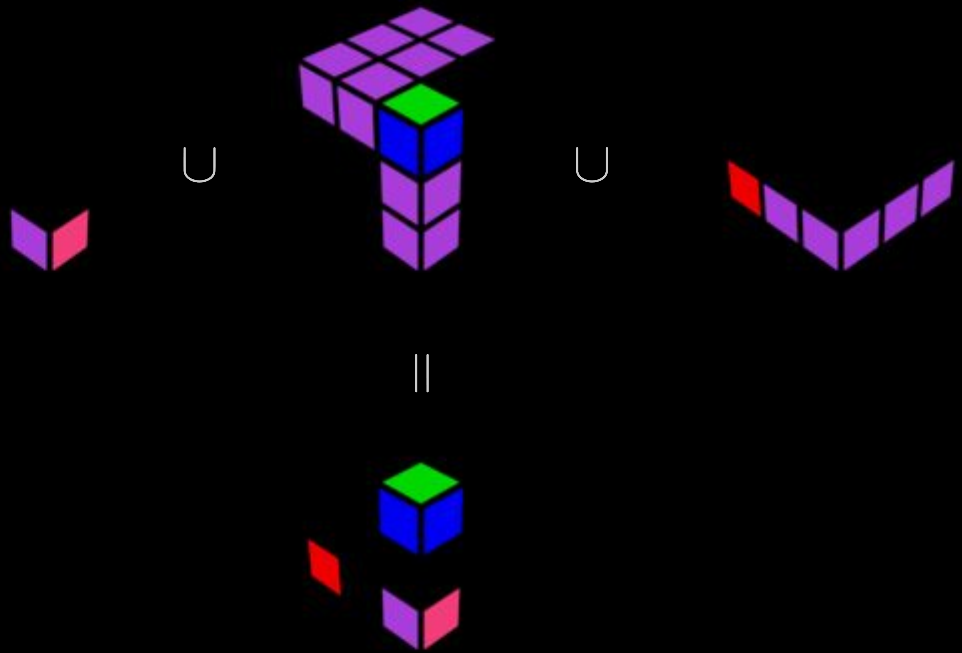
$$\mathrm{mov}([X, Y]) \subseteq \mathrm{mov}(X, Y) \cup X' \,\mathrm{mov}(X, Y) \cup Y' \,\mathrm{mov}(X, Y)$$

commutators

[R U R′, D]

X

Y



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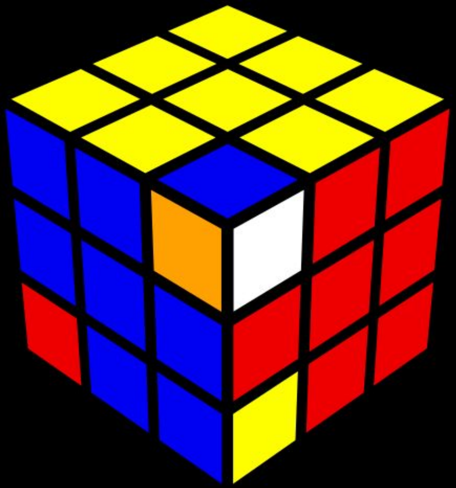
commutators

[R U R′, D]

X

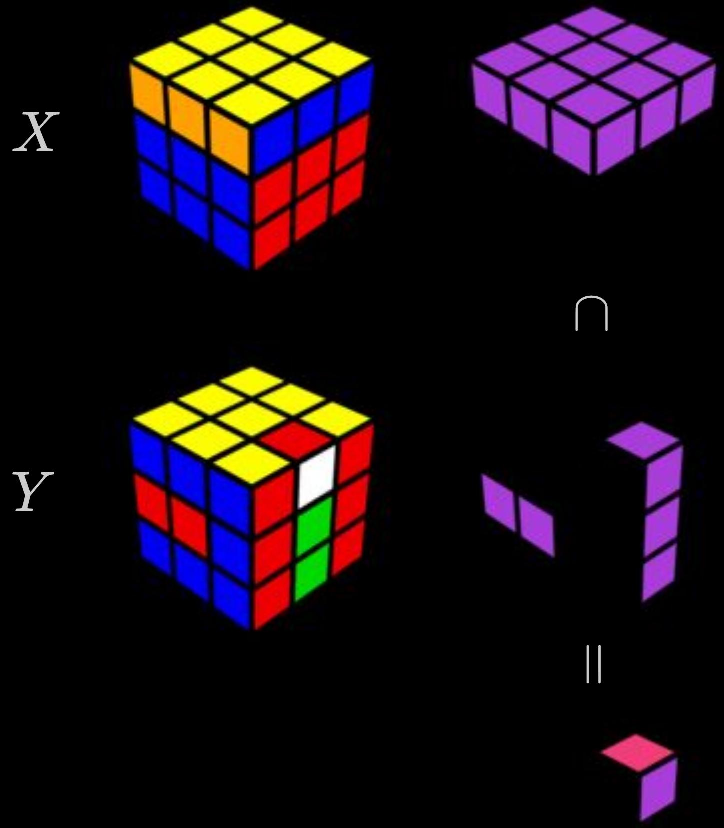
Y



$$\mathrm{mov}([X, Y]) \subseteq \mathrm{mov}(X, Y) \cup X' \, \mathrm{mov}(X, Y) \cup Y' \, \mathrm{mov}(X, Y)$$

commutators

[U', R' E' R]



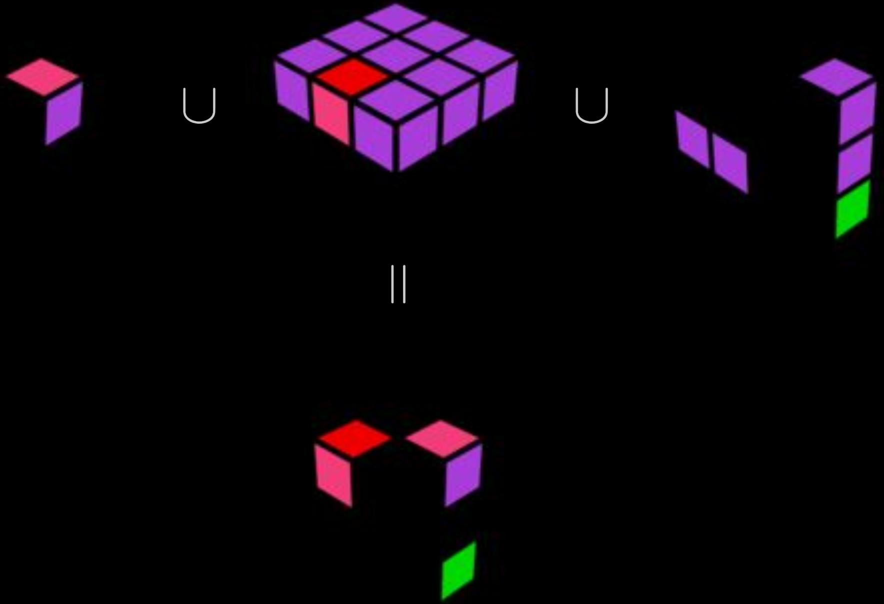
$$\mathrm{mov}([X,Y]) \subseteq \mathrm{mov}(X,Y) \cup X' \, \mathrm{mov}(X,Y) \cup Y' \, \mathrm{mov}(X,Y)$$

commutators

[U', R' E' R]

X

Y



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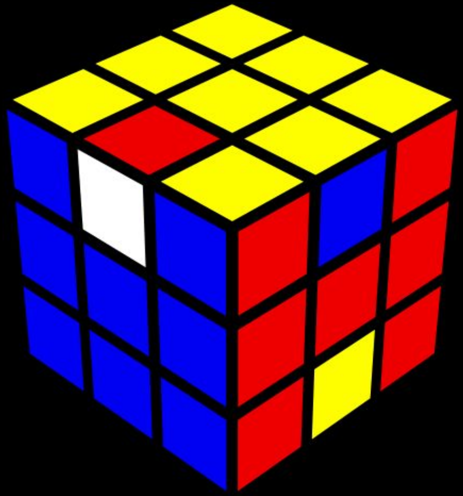
commutators

[U’, R’ E’ R]

X

Y



BH/3-cycles/3-style method (good name)

solve cube by applying conjugated commutators that cycle 3 pieces to solve individual pieces.

every 3 cycle must involve some constant ‘buffer’ piece.

this is a blindfolded method! you can figure out and memorize the cycles needed without applying moves.

$$r = \sigma_1 \sigma_2 \ldots \sigma_n$$

$$\tau_n = (a\, b\, c)$$

a is the buffer, b and c are the other pieces

$$\tau_1 \tau_2 \ldots \tau_n r = e$$

Example

This method cannot fix 2e2c permutation parity! You must fix it separately

stuff we couldn't fit in

- 2x2, 4x4 cubes and larger
  - centre parity, problems when reducing to 3x3x3
- fewest moves competition
  - NISS, insertions, domino reduction
- Thistlewaite's / Kociemba's algorithms
  - reduce to successively smaller subgroups
  - lookup table exploiting cube symmetry
- God's number
  - computer-assisted coset enumeration of DR subgroup

resources

permutation puzzles textbook

some elementary subgroups of the cube group

cube group conjugacy classes (hard!)

theory of semidirect products (hard!!!)

why we need semidirect product for the cube group

References

[1] 2x2x2 Cayley graph: https://miscellaneouscoder.wordpress.com/2014/07/28/working-with-rubiks-group-cycle-graphs/

[2] Rubik’s Cube 3D model: https://grabcad.com/library/rubik-s-cube-36