The Ghost Algebra

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Outline

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2. What is the ghost algebra?
3. What is the dilute ghost algebra?
4. What can we do with these algebras?

QUESTION 1:

Why do we want a new algebra?

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and **unital**, meaning there exists $I \in A$ such that

$$Ia = aI = a, \qquad \forall a \in A.$$

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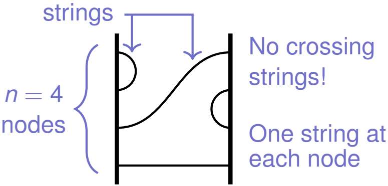

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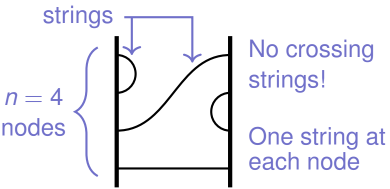

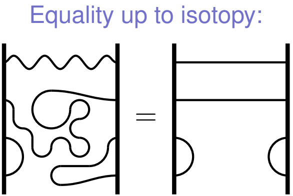

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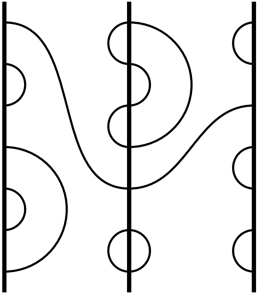

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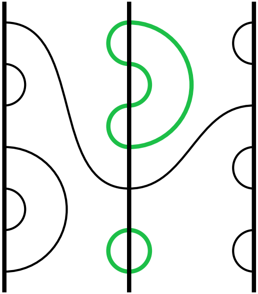

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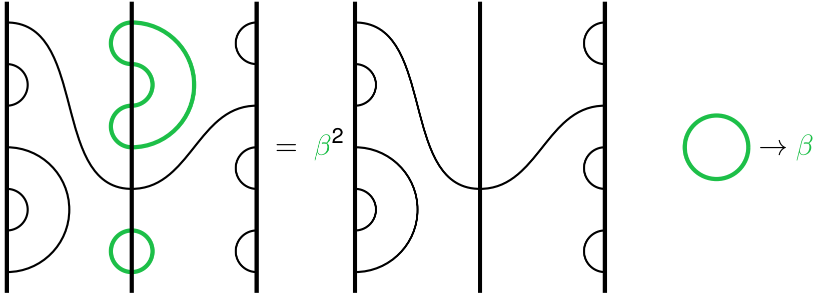

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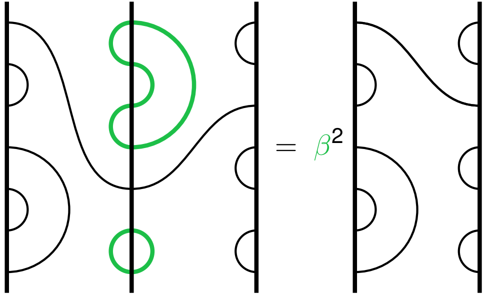

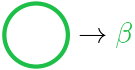

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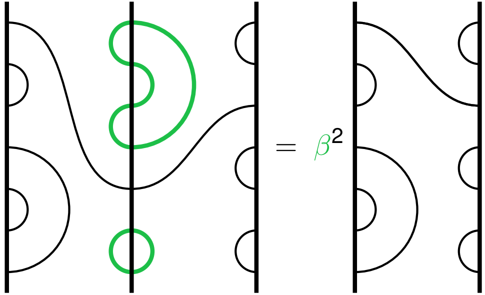

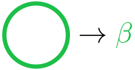

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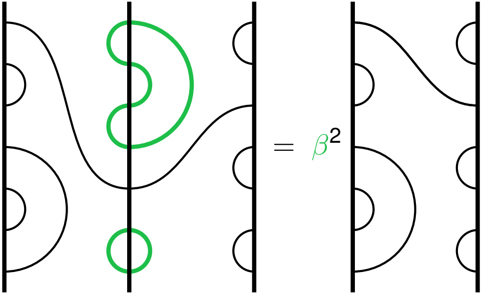

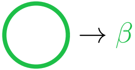

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- Dimensions given by Catalan numbers

One-boundary TL algebra $\mathrm{TL}_n^1(\beta; \alpha_1, \alpha_2)$

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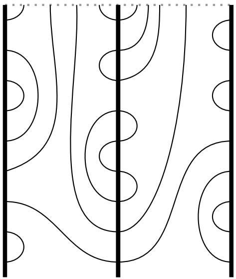

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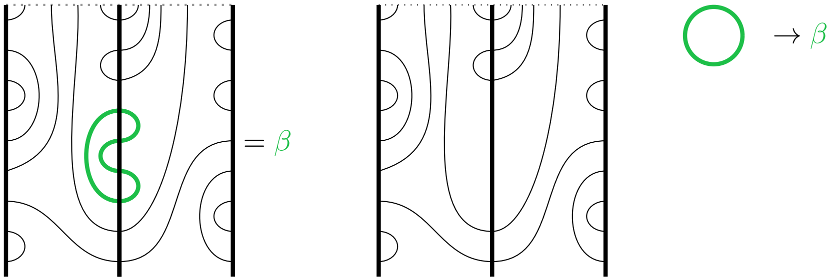

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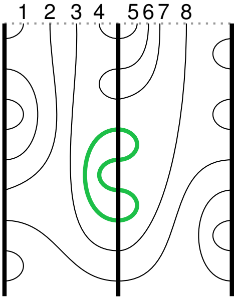

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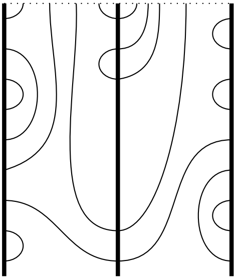

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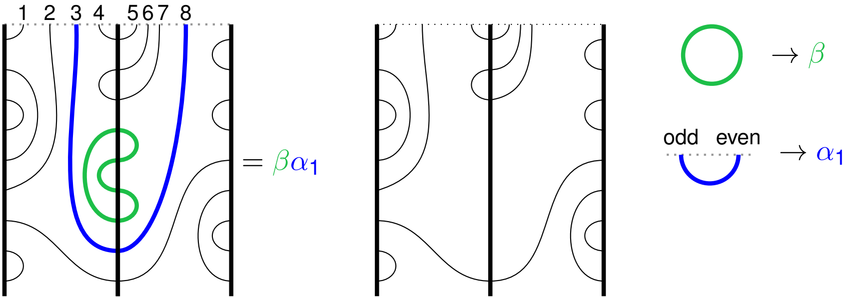

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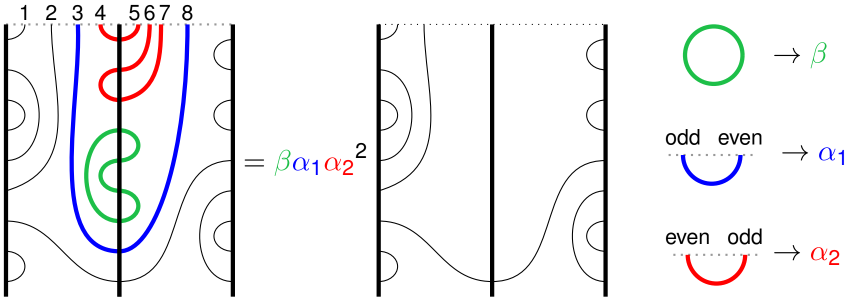

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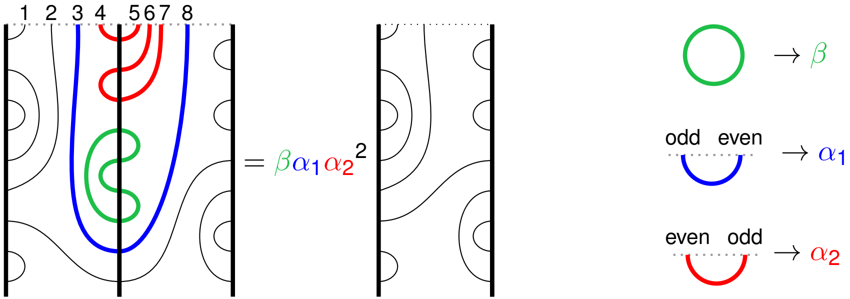

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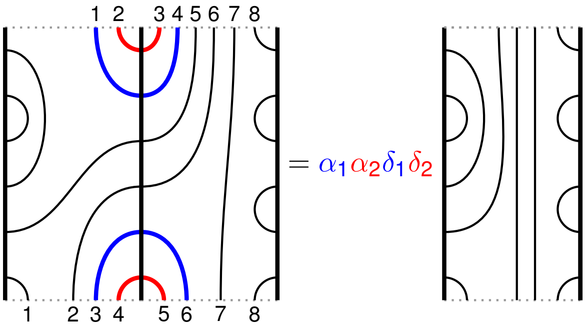

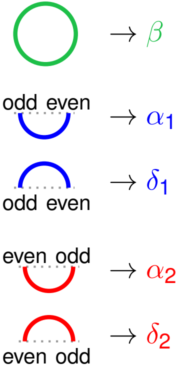

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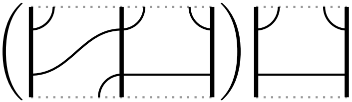

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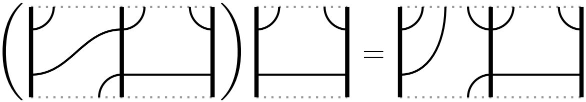

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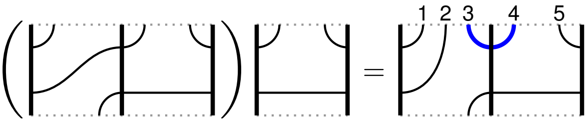

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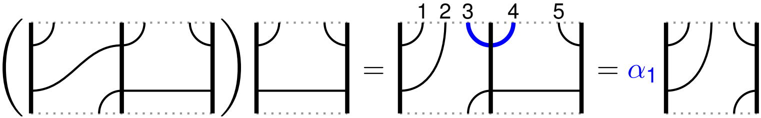

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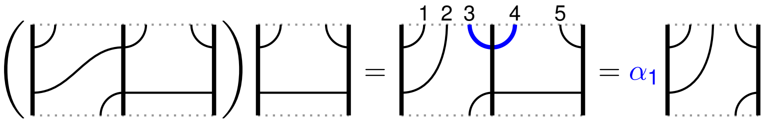

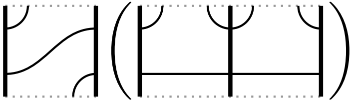

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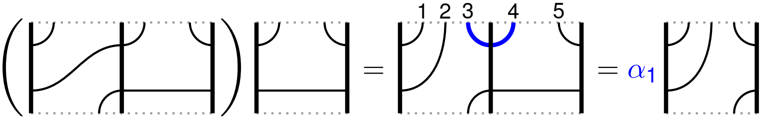

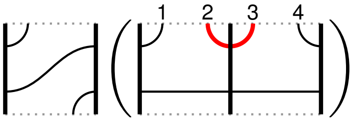

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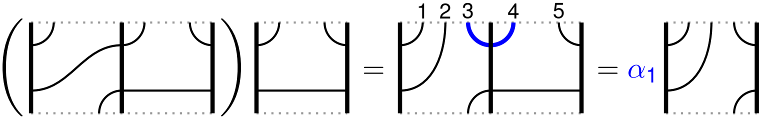

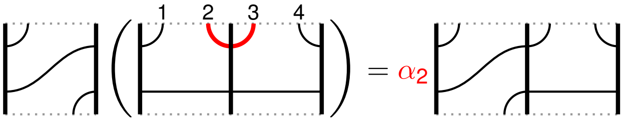

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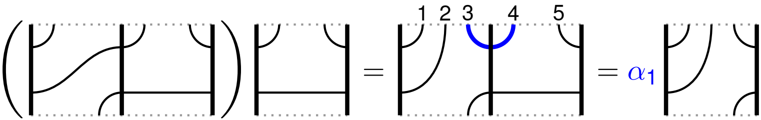

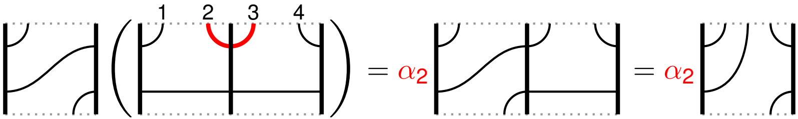

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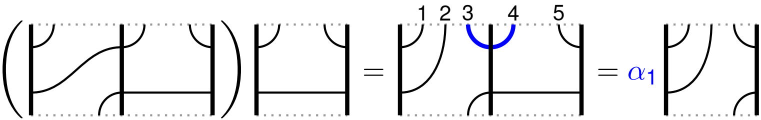

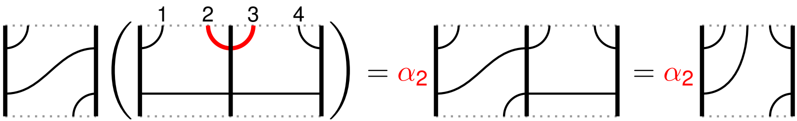

- Associativity implies $\alpha_1 = \alpha_2$

QUESTION 2:

What is the ghost algebra?

The ghost algebra Gh_n^2

Basis of two-boundary *n*-diagrams with **ghosts**: dots on boundaries that keep track of string parity

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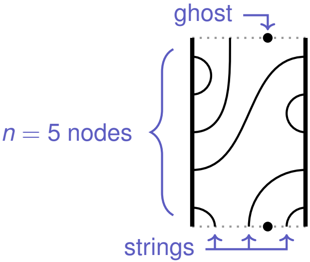

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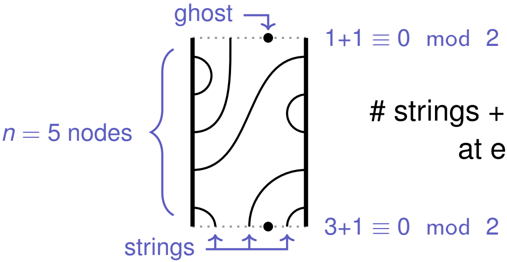

strings + # ghosts $\equiv 0 \mod 2$
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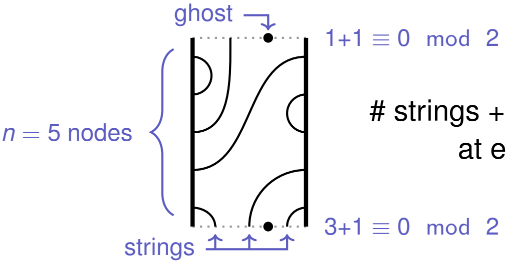
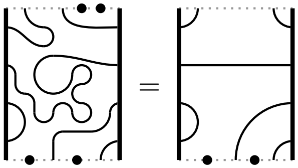


strings + # ghosts $\equiv 0 \mod 2$ at each boundary

Equality: string isotopy; same # ghosts in each boundary region mod 2



Ghost algebra multiplication

Boundary-to-boundary strings leave a ghost behind at each endpoint

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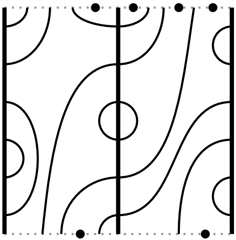

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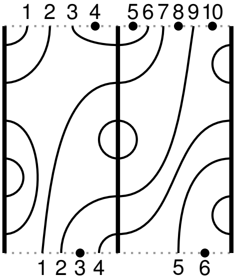

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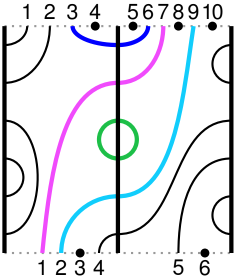

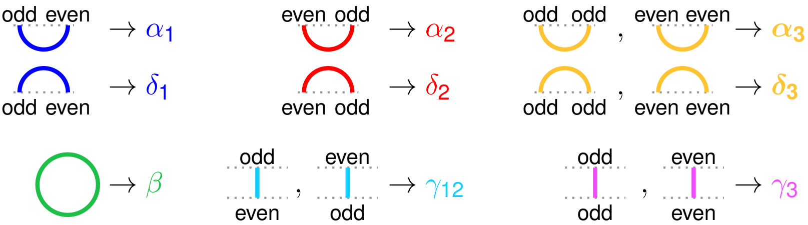

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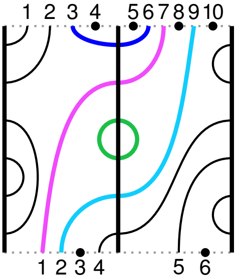
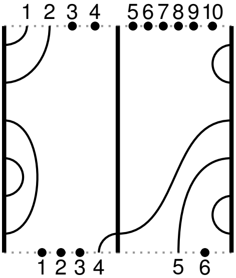
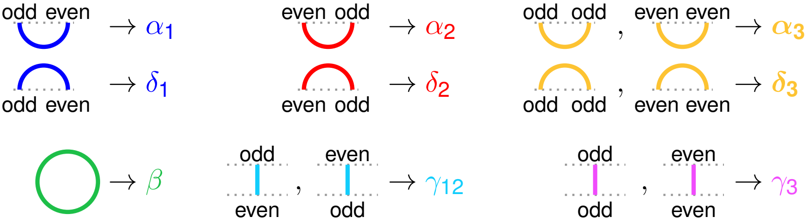


$= \beta \alpha_1 \gamma_{12} \gamma_3$





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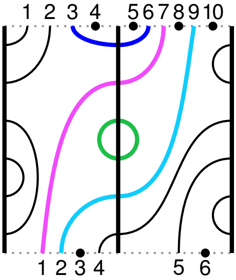

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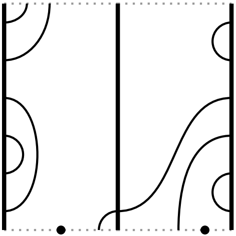

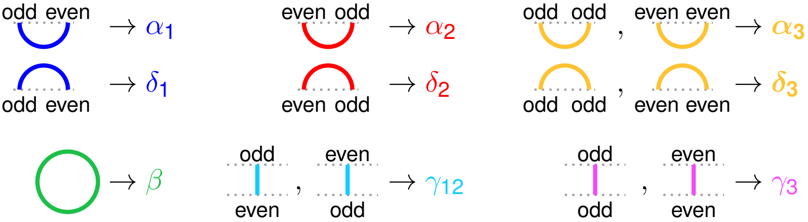

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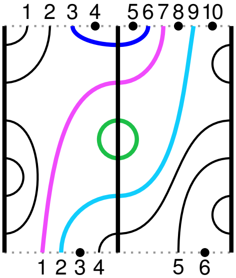

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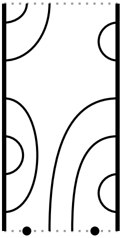

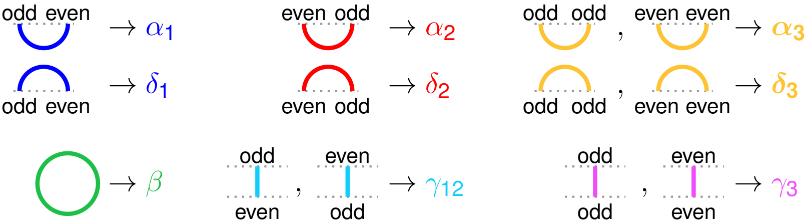

Evenness condition in TL_n^2

Why could TL_n^2 require an even number of strings at each boundary?

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What if some nodes didn't have any strings connected to them?

QUESTION 3:

What is the dilute ghost algebra?

Dilute TL algebra

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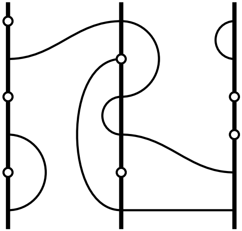 $= 0,$

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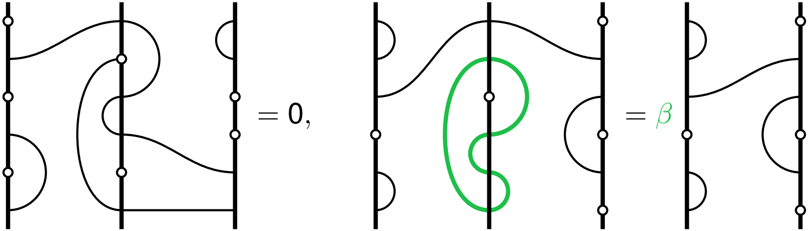

Boundary dilute algebras

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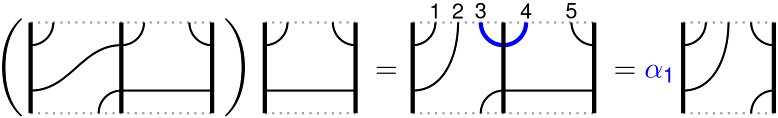

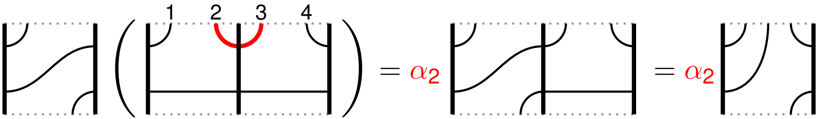

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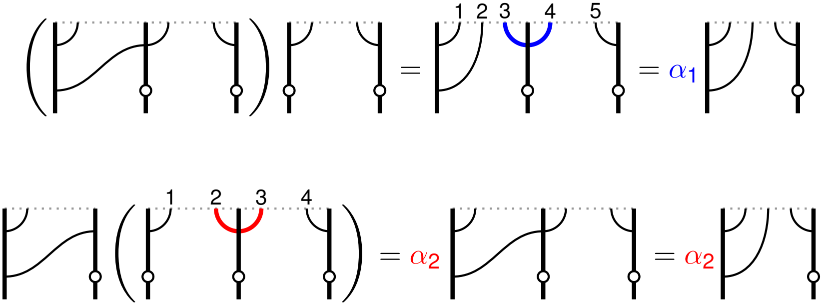

Boundary associativity problem occurs even in one-boundary case.

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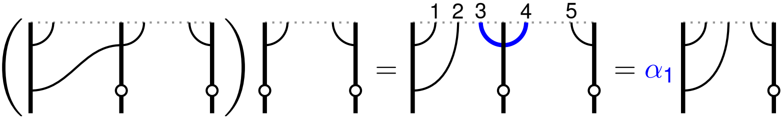

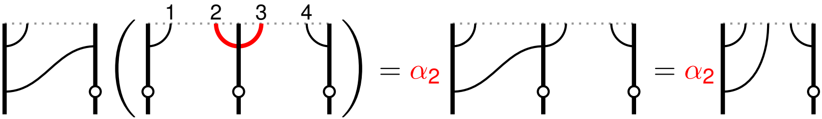

Boundary associativity problem occurs even in one-boundary case.

No algebras in literature have $\mathrm{dTL}_n(\beta)$ and $\mathrm{TL}_n^1(\beta; \alpha_1, \alpha_2)$ as subalgebras with $\alpha_1 \neq \alpha_2$.

Dilute ghost algebra dGh_n^2

Dilute ghost algebra dGh_n^2 allows n-diagrams with empty nodes

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Dilute ghost algebra dGh_n^2 allows *n*-diagrams with empty nodes

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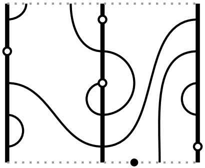

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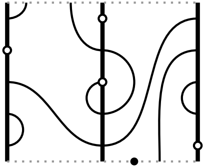 $= 0,$

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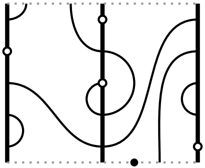

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Same parameters as Gh_n^2:

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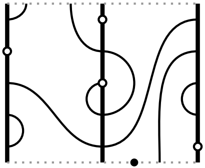

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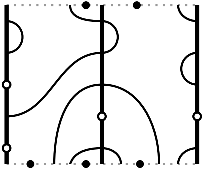

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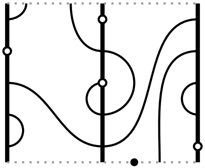

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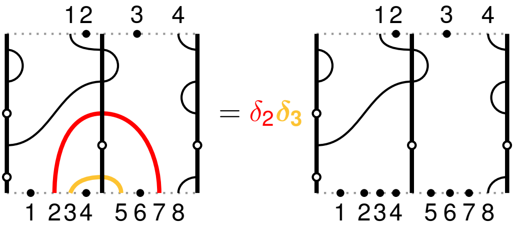

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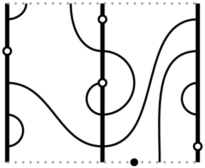

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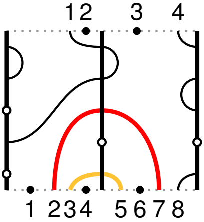

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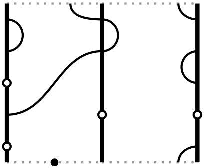

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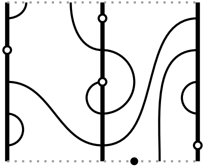

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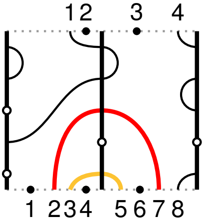

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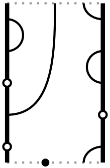

Algebra inclusions

$$\mathrm{Gh}_n^2\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta & \gamma_{12} & \gamma_3\\ \delta_1 & \delta_2 & \delta_3\end{pmatrix}$$

Algebra inclusions

$$\mathrm{Gh}_n^1(\beta; \alpha_1, \alpha_2, \alpha_3) \hookrightarrow \mathrm{Gh}_n^2\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta & \gamma_{12} & \gamma_3 \\ \delta_1 & \delta_2 & \delta_3 \end{pmatrix}$$

Algebra inclusions

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\mathrm{TL}_n^1(\beta;\alpha_1,\alpha_2) & & \\
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Algebra inclusions

$$\begin{array}{ccccc}
\mathrm{TL}_n(\beta) & \hookrightarrow & \mathrm{TL}^1_n(\beta;\alpha_1,\alpha_2) & & \\
 & & \big\downarrow & & \\
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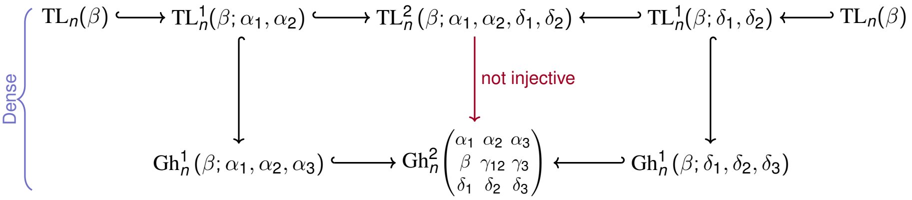
Algebra inclusions



Algebra inclusions

Dense

$$\mathrm{TL}_n(\beta) \hookrightarrow \mathrm{TL}_n^1(\beta;\alpha_1,\alpha_2) \hookrightarrow \mathrm{TL}_n^2(\beta;\alpha_1,\alpha_2,\delta_1,\delta_2) \hookleftarrow \mathrm{TL}_n^1(\beta;\delta_1,\delta_2) \hookleftarrow \mathrm{TL}_n(\beta)$$

$$\mathrm{TL}_n^1(\beta;\alpha_1,\alpha_2) \hookrightarrow \mathrm{Gh}_n^1(\beta;\alpha_1,\alpha_2,\alpha_3), \qquad \mathrm{TL}_n^2(\beta;\alpha_1,\alpha_2,\delta_1,\delta_2) \xrightarrow{\text{not injective}} \mathrm{Gh}_n^2\begin{pmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta & \gamma_{12} & \gamma_3\\ \delta_1 & \delta_2 & \delta_3\end{pmatrix}, \qquad \mathrm{TL}_n^1(\beta;\delta_1,\delta_2) \hookrightarrow \mathrm{Gh}_n^1(\beta;\delta_1,\delta_2,\delta_3)$$

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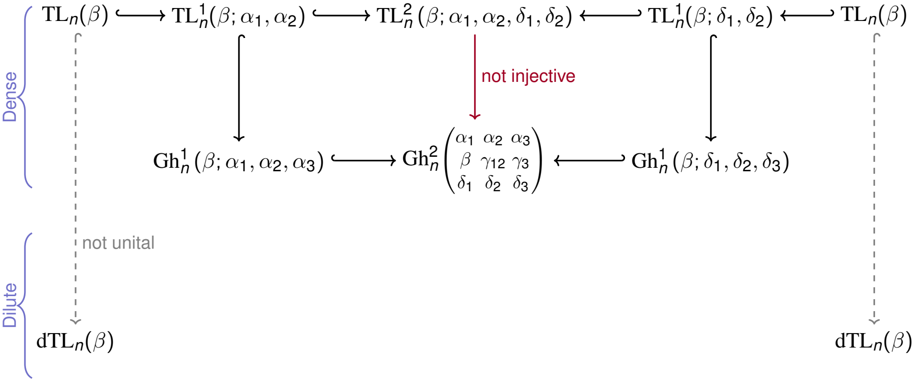
Dilute

$$\mathrm{dTL}_n(\beta) \qquad\qquad \mathrm{dTL}_n(\beta)$$

Algebra inclusions



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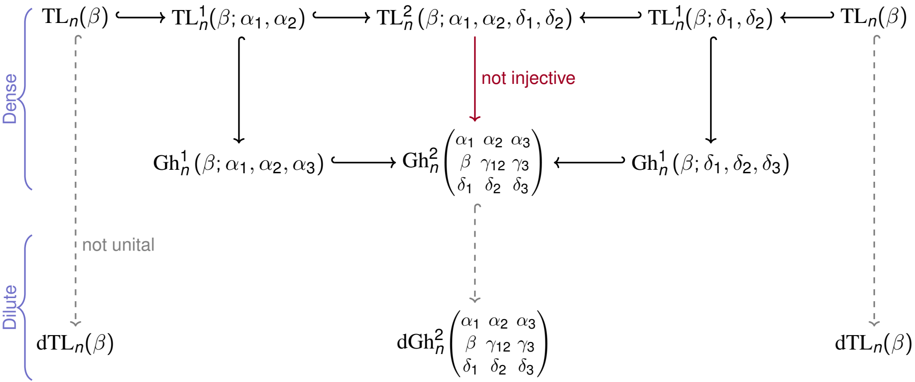

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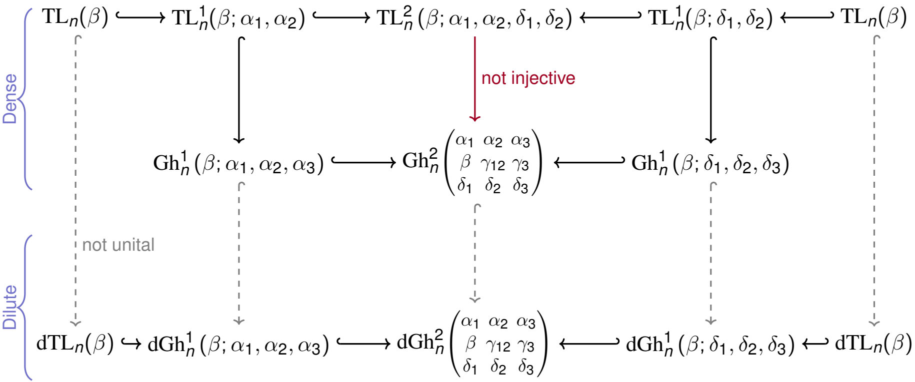

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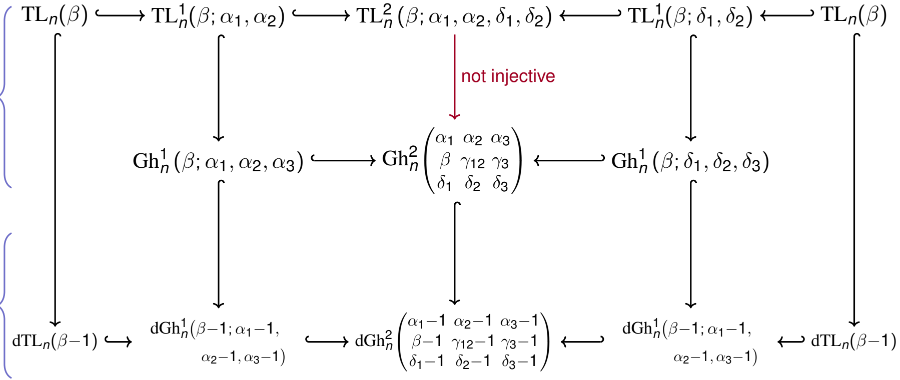
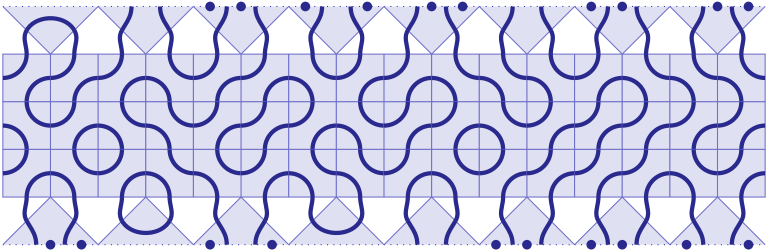


QUESTION 4:

What can we do with these algebras?

Loop model

Build a lattice out of **bulk squares** and **boundary triangles**



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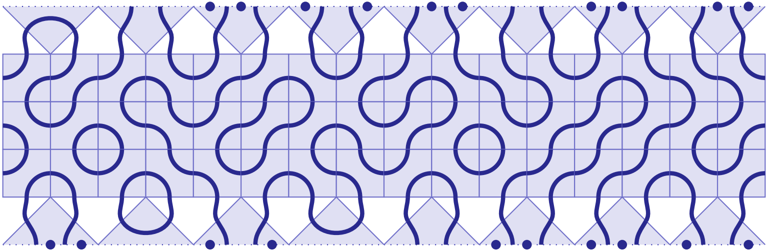

Construct Hamiltonians using solutions to **Yang-Baxter equation (YBE)** and **boundary Yang-Baxter equation (BYBE)**.

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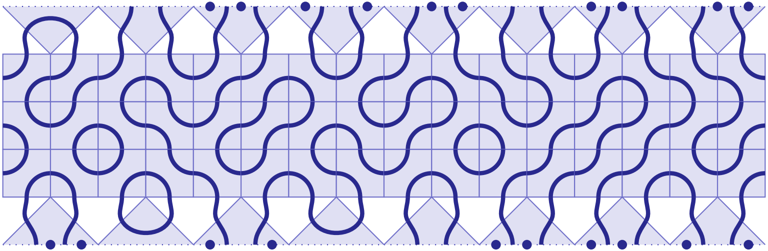

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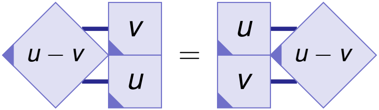
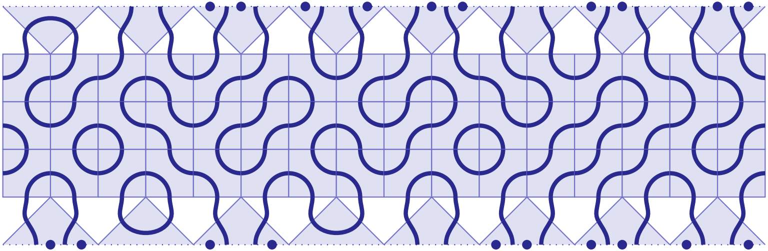
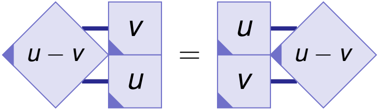


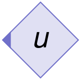
Build a lattice out of **bulk squares** and **boundary triangles**



Construct Hamiltonians using solutions to **Yang-Baxter equation (YBE)** and **boundary Yang-Baxter equation (BYBE)**.

YBE:





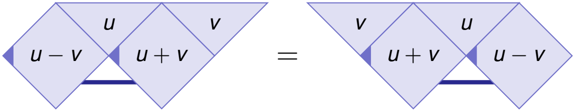
$$:= \frac{\sin(\lambda - u)}{\sin(\lambda)}$$

$$+ \frac{\sin(u)}{\sin(\lambda)}$$

$$, \qquad \beta = 2\cos\lambda$$

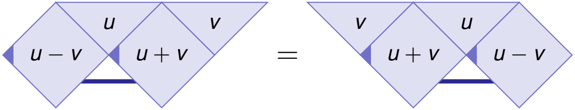
Boundary Yang-Baxter equation

The BYBE involves **boundary face operators**



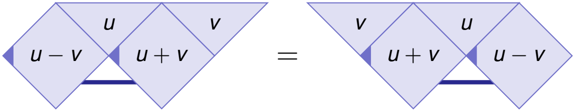
Boundary Yang-Baxter equation

The BYBE involves **boundary face operators**



Two solutions: $\rho \in \mathbb{C}$, $\beta = 2\cos\lambda$, $b : \mathbb{C} \to \mathbb{C}$,

The BYBE involves **boundary face operators**



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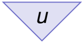
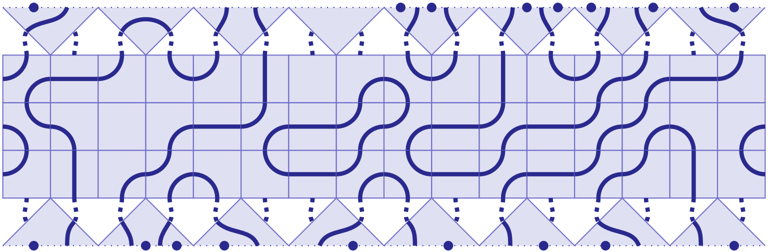


$= b(u) \Big(\frac{F(u)}{\sin(2u)}$  $+ c_1$  $+ c_2$  $+ c_3$  $+ c_4$  $\Big)$

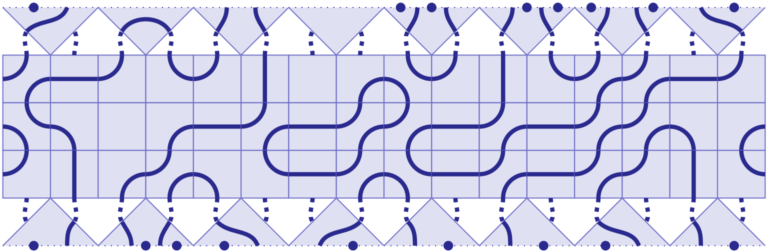
$$\begin{cases} F(u) = \rho + (c_1\alpha_1 + c_3\alpha_2 + (c_2 + c_4)\alpha_3)\cos(2u) \\ \qquad\qquad -(c_3\alpha_1 + c_1\alpha_2 + (c_2 + c_4)\alpha_3)\cos(2u - \lambda), \\ c_1 c_3 = c_2 c_4, \end{cases}$$

$$\begin{cases} F(u) = \rho + \left(2\alpha_3^2 - \alpha_1^2 - \alpha_2^2\right)\cos(2u) - \left(\alpha_3^2 - \alpha_1\alpha_2\right)\cos(2u - \lambda), \\ c_1 = -\alpha_1, \qquad c_2 = c_4 = \alpha_3, \qquad c_3 = -\alpha_2. \end{cases}$$

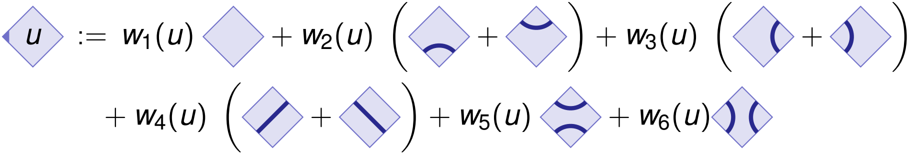
Dilute loop model



Dilute loop model



Batchelor & Yung (1995) — YBE solution ($\beta = -2\cos(4\phi)$)

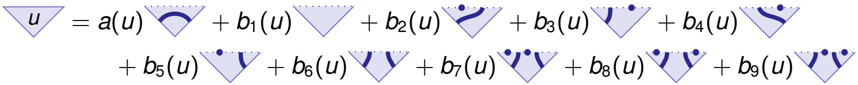


$$w_1(u) = \sin(2\phi)\sin(3\phi) + \sin u \sin(3\phi - u) \qquad w_2(u) = \sin(2\phi)\sin(3\phi - u)$$

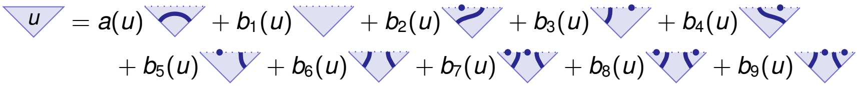
$$w_3(u) = \sin(2\phi)\sin u \qquad w_4(u) = \sin u \sin(3\phi - u)$$

$$w_5(u) = \sin(2\phi - u)\sin(3\phi - u) \qquad w_6(u) = -\sin u \sin(\phi - u)$$

Dilute ghost algebra BYBE solution



Dilute ghost algebra BYBE solution



$$a(u) = \frac{-f(u)\sin\left(u - \frac{3\phi}{2}\right)}{\sin(2u)\sin(2\phi)}\left(2\rho^2\cos(\phi) - \kappa^2\left(\left(\mu^2+\nu^2\right)\alpha_3 + \mu\nu(\alpha_1+\alpha_2)\right)\sin^2\left(u - \frac{\phi}{2}\right)\right)$$

$$b_1(u) = \frac{f(u)}{\sin(2u)\sin(2\phi)}\left(2\rho^2\sin\left(u + \frac{3\phi}{2}\right)\cos(\phi) - \kappa^2\left(\left(\mu^2+\nu^2\right)\alpha_3 + \mu\nu(\alpha_1+\alpha_2)\right)\sin\left(u+\frac{\phi}{2}\right)\sin\left(u-\frac{\phi}{2}\right)\sin\left(u-\frac{3\phi}{2}\right)\right)$$

$$b_2(u) = b_5(u) = \nu\kappa\rho f(u), \qquad b_3(u) = b_4(u) = \mu\kappa\rho f(u)$$

$$b_6(u) = b_8(u) = \mu\nu\kappa^2 f(u)\sin\left(u - \frac{\phi}{2}\right)$$

$$b_7(u) = \nu^2\kappa^2 f(u)\sin\left(u - \frac{\phi}{2}\right), \qquad b_9(u) = \mu^2\kappa^2 f(u)\sin\left(u - \frac{\phi}{2}\right)$$

$$a(u) = \frac{f(u)\cos\left(u - \frac{3\phi}{2}\right)}{\sin(2u)\sin(2\phi)}\left(2\rho^2\cos(\phi) - \kappa^2\left(\left(\mu^2+\nu^2\right)\alpha_3 - \mu\nu(\alpha_1+\alpha_2)\right)\cos^2\left(u - \frac{\phi}{2}\right)\right)$$

$$b_1(u) = \frac{f(u)}{\sin(2u)\sin(2\phi)}\left(2\rho^2\cos\left(u + \frac{3\phi}{2}\right)\cos(\phi) - \kappa^2\left(\left(\mu^2+\nu^2\right)\alpha_3 - \mu\nu(\alpha_1+\alpha_2)\right)\cos\left(u+\frac{\phi}{2}\right)\cos\left(u-\frac{\phi}{2}\right)\cos\left(u-\frac{3\phi}{2}\right)\right)$$

$$b_2(u) = \nu\kappa\rho f(u), \qquad b_3(u) = -\tau\kappa\rho f(u), \qquad b_4(u) = \tau\kappa\rho f(u), \qquad b_5(u) = -\nu\kappa\rho f(u)$$

$$b_6(u) = b_8(u) = -\mu\nu\kappa^2 f(u)\cos\left(u - \frac{\phi}{2}\right)$$

$$b_7(u) = \nu^2\kappa^2 f(u)\cos\left(u - \frac{\phi}{2}\right), \qquad b_9(u) = \mu^2\kappa^2 f(u)\cos\left(u - \frac{\phi}{2}\right)$$

$$a(u) = \frac{-b(u)}{2\sin(2u)}\,\frac{\left(\alpha_1^2+\alpha_2^2-2\alpha_3^2\right)(\cos(u) - k\sin(u))(\cos(u-\phi) - k\sin(u-\phi))}{\left(k\sin\left(\frac{3\phi}{2}\right) - \cos\left(\frac{3\phi}{2}\right)\right)\left(k\cos\left(\frac{3\phi}{2}\right) + \sin\left(\frac{3\phi}{2}\right)\right)}$$

$$b_1(u) = \frac{-b(u)}{2\sin(2u)}\,\frac{\left(\alpha_1^2+\alpha_2^2-2\alpha_3^2\right)(\cos(u) + k\sin(u))(\cos(u-\phi) - k\sin(u-\phi))}{\left(k\sin\left(\frac{3\phi}{2}\right) - \cos\left(\frac{3\phi}{2}\right)\right)\left(k\cos\left(\frac{3\phi}{2}\right) + \sin\left(\frac{3\phi}{2}\right)\right)}$$

$$b_2(u) = b_3(u) = b_4(u) = b_5(u) = 0$$

$$b_6(u) = -\alpha_1 b(u), \qquad b_7(u) = b_9(u) = \alpha_3 b(u), \qquad b_8(u) = -\alpha_2 b(u)$$

$$\left(k^2-1\right)\left(\left(\alpha_1\alpha_2 - \alpha_3^2\right)\sin(3\phi) - \left(\alpha_1^2+\alpha_2^2-2\alpha_3^2\right)\sin(\phi)\right) = 2k\left(\left(\alpha_1\alpha_2-\alpha_3^2\right)\cos(3\phi) + \left(\alpha_1^2+\alpha_2^2-2\alpha_3^2\right)\cos(\phi)\right)$$

$$a(u) = \frac{-b(u)}{\sin(2u)}(\alpha_1-\alpha_2)\left(\mu\nu(\alpha_1+\alpha_2) + \left(\mu^2+\nu^2\right)\alpha_3\right)(\sigma\cos(u) - \tau\sin(u))(\sigma\cos(u-\phi) - \tau\sin(u-\phi))$$

$$b_1(u) = \frac{-b(u)}{\sin(2u)}(\alpha_1-\alpha_2)\left(\mu\nu(\alpha_1+\alpha_2) + \left(\mu^2+\nu^2\right)\alpha_3\right)(\sigma\cos(u) + \tau\sin(u))(\sigma\cos(u-\phi) - \tau\sin(u-\phi))$$

$$b_2(u) = b_3(u) = b_4(u) = b_5(u) = 0$$

$$b_6(u) = \mu b(u)\left((\nu\alpha_2+\mu\alpha_3)\left(2\sigma\tau\cos(\phi) - \left(\sigma^2-\tau^2\right)\sin(\phi)\right) + (\nu\alpha_1+\mu\alpha_3)\left(2\sigma\tau\cos(3\phi) + \left(\sigma^2-\tau^2\right)\sin(3\phi)\right)\right)$$

$$b_7(u) = \nu b(u)\left((\nu\alpha_2+\mu\alpha_3)\left(2\sigma\tau\cos(\phi) - \left(\sigma^2-\tau^2\right)\sin(\phi)\right) + (\nu\alpha_1+\mu\alpha_3)\left(2\sigma\tau\cos(3\phi) + \left(\sigma^2-\tau^2\right)\sin(3\phi)\right)\right)$$

$$b_8(u) = -\nu b(u)\left((\mu\alpha_1+\nu\alpha_3)\left(2\sigma\tau\cos(\phi) - \left(\sigma^2-\tau^2\right)\sin(\phi)\right) + (\mu\alpha_2+\nu\alpha_3)\left(2\sigma\tau\cos(3\phi) + \left(\sigma^2-\tau^2\right)\sin(3\phi)\right)\right)$$

$$b_9(u) = -\mu b(u)\left((\mu\alpha_1+\nu\alpha_3)\left(2\sigma\tau\cos(\phi) - \left(\sigma^2-\tau^2\right)\sin(\phi)\right) + (\mu\alpha_2+\nu\alpha_3)\left(2\sigma\tau\cos(3\phi) + \left(\sigma^2-\tau^2\right)\sin(3\phi)\right)\right)$$

$$a(u) = \frac{(c_6\alpha_1 + c_8\alpha_2 + (c_7+c_9)\alpha_3)b(u)(\sigma\cos(u-\phi) + \tau\sin(u-\phi))(\sigma\cos u + \tau\sin u)}{\sin(2u)\left(2\sigma\tau\cos(3\phi) - (\sigma^2-\tau^2)\sin(3\phi)\right)}$$

$$b_1(u) = \frac{(c_6\alpha_1 + c_8\alpha_2 + (c_7+c_9)\alpha_3)b(u)(\sigma\cos(u-\phi) + \tau\sin(u-\phi))(\sigma\cos u - \tau\sin u)}{\sin(2u)\left(2\sigma\tau\cos(3\phi) - (\sigma^2-\tau^2)\sin(3\phi)\right)}$$

$$b_2(u) = b_3(u) = b_4(u) = b_5(u) = 0$$

$$b_6(u) = c_6 b(u), \qquad b_7(u) = c_7 b(u), \qquad b_8(u) = c_8 b(u), \qquad b_9(u) = c_9 b(u)$$

$$c_6c_8 = c_7c_9, \qquad A_1 = \alpha_1^2+\alpha_2^2-2\alpha_3^2, \qquad A_2 = \alpha_1\alpha_2 - \alpha_3^2$$

$$(c_6\alpha_1 + c_8\alpha_2 + (c_7+c_9)\alpha_3)\left(2\sigma\tau\cos\phi + \left(\sigma^2-\tau^2\right)\sin\phi\right) + (c_8\alpha_1 + c_6\alpha_2 + (c_7+c_9)\alpha_3)\left(2\sigma\tau\cos(3\phi) - \left(\sigma^2-\tau^2\right)\sin(3\phi)\right)$$

$$(c_7\alpha_2 + c_6\alpha_3)\left(2\sigma\tau\cos\phi + \left(\sigma^2-\tau^2\right)\sin\phi\right) + (c_7\alpha_1 + c_6\alpha_3)\left(2\sigma\tau\cos(3\phi) - \left(\sigma^2-\tau^2\right)\sin(3\phi)\right)$$

$$\left(\sigma^2+\tau^2\right)^2(2A_2 + A_1\cos(4\phi)) - \left(\sigma^4 - 6\sigma^2\tau^2 + \tau^4\right)((A_1+A_2)\cos(2\phi) + A_2\cos(6\phi)) - 4\sigma\tau\left(\sigma^2-\tau^2\right)((A_1-A_2)\sin(2\phi) + A_2\sin(6\phi)) = 0$$

What else can we do with these algebras?

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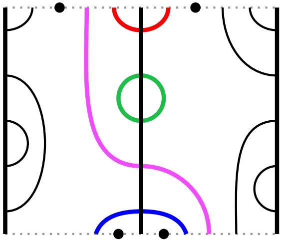
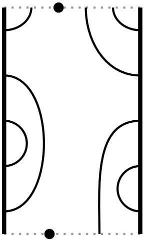


$= \beta \alpha_2 \gamma_3 \delta_1$



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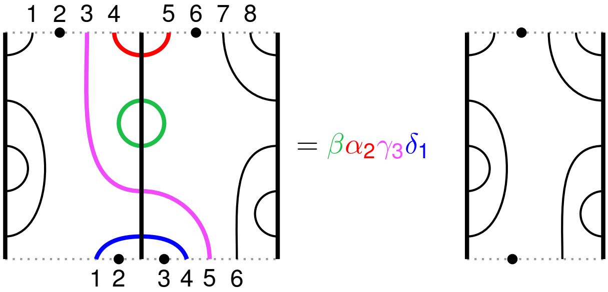

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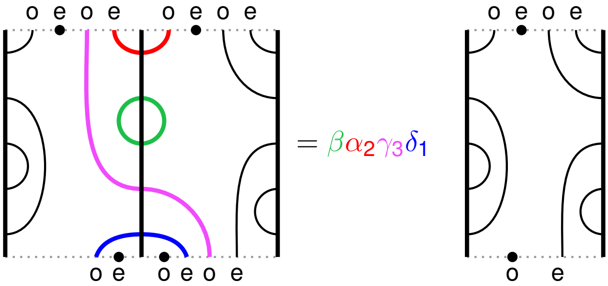

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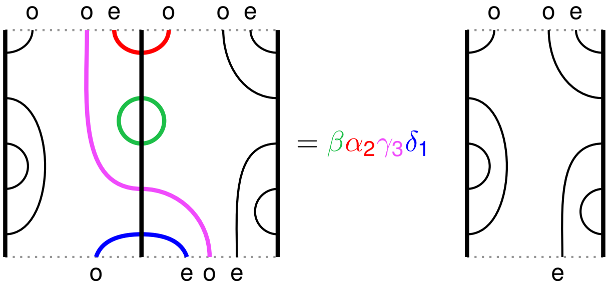
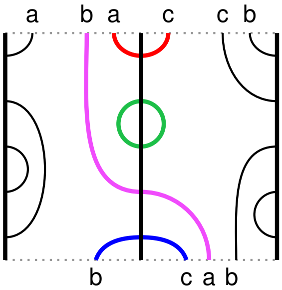
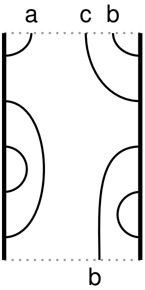


What else can we do with these algebras?

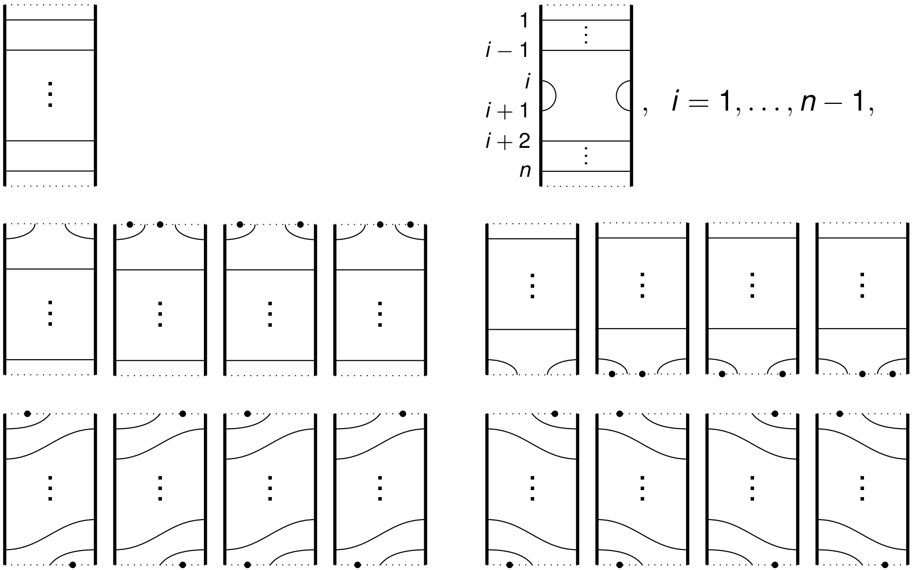
- Representation theory — use cellularity
- Physics — Hamiltonians and energy eigenvalues
- Further generalisations — **label algebra**



$= \beta\alpha^{ac}\gamma^{b}_{a}\delta_{bc}$



Ghost algebra generators



Dimensions

$$\dim \mathrm{Gh}_n^1 = \sum_{d=0}^{n} \left(\sum_{j=0}^{\lfloor \frac{n-d}{2} \rfloor} 2^{n-2j-d} \left(\binom{n}{j} - \binom{n}{j-1} \right) \right)^2$$

$$\dim \mathrm{Gh}_n^2 = \sum_{d=0}^{n} \left(\sum_{j=0}^{\lfloor \frac{n-d}{2} \rfloor} 2^{n-2j-d}(n-2j-d+1) \left(\binom{n}{j} - \binom{n}{j-1} \right) \right)^2$$

$$\dim \mathrm{dGh}_n^1 = \sum_{d=0}^{n} \left(\sum_{v=0}^{n-d} \binom{n}{v} \sum_{j=0}^{\lfloor \frac{n-v-d}{2} \rfloor} 2^{n-v-2j-d} \left(\binom{n-v}{j} - \binom{n-v}{j-1} \right) \right)^2$$

$$\dim \mathrm{dGh}_n^2 = \sum_{d=0}^{n} \left(\sum_{v=0}^{n-d} \binom{n}{v} \sum_{j=0}^{\lfloor \frac{n-v-d}{2} \rfloor} 2^{n-v-2j-d} (n-v-2j-d+1) \times \left(\binom{n-v}{j} - \binom{n-v}{j-1} \right) \right)^2$$

Table of dimensions

n	$\dim \mathrm{Gh}^1_n$	$\dim \mathrm{Gh}^2_n$	$\dim \mathrm{dGh}^1_n$	$\dim \mathrm{dGh}^2_n$
1	5	17	10	26
2	30	186	117	521
3	185	1,813	1,407	9,355
4	1,150	16,102	17,083	156,947
5	7,170	135,866	208,284	2,514,932
6	44,760	1,099,276	2,544,751	38,968,815
7	279,585	8,639,133	31,125,138	588,475,298
8	1,746,870	66,258,526	380,928,795	8,706,799,523
9	10,916,150	498,701,470	4,663,705,782	126,690,947,758
10	68,219,860	3,693,607,300	57,109,857,519	1,818,028,127,339