

The Ghost Algebra



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The University of Queensland

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12 May 2023

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- 1 Why do we want a new algebra?
- 2 What is the ghost algebra?
- 3 What is the dilute ghost algebra?
- 4 What can we do with these algebras?

QUESTION 1:

Why do we want a new algebra?

What is an algebra?

Think: $n \times n$ matrices over a field \mathbb{F}

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$$a(\lambda b + \mu c) = \lambda ab + \mu ac$$

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$$a(bc) = (ab)c, \quad \forall a, b, c \in A$$

and **unital**, meaning there exists $1 \in A$ such that

$$1a = a1 = a, \quad \forall a \in A.$$

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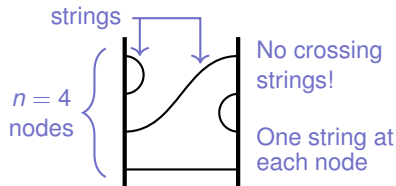
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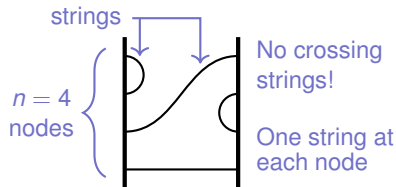
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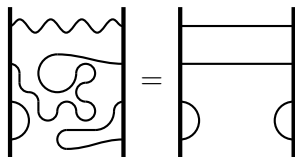


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Equality up to isotopy:



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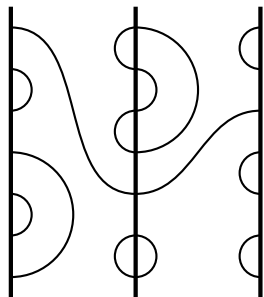
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- Fix $\beta \in \mathbb{C}$
- Multiplication defined on n -diagrams, extended bilinearly

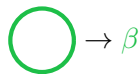
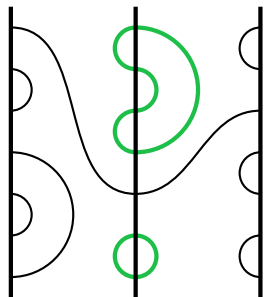
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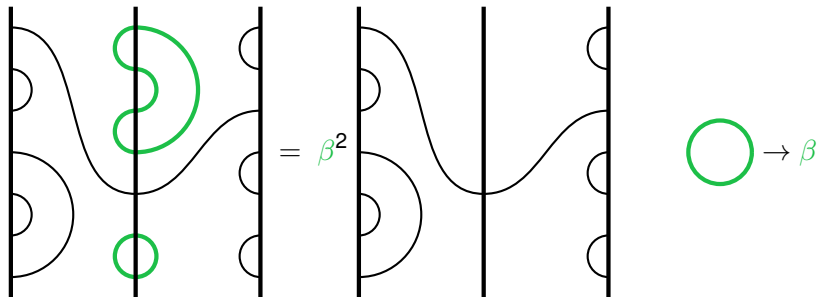
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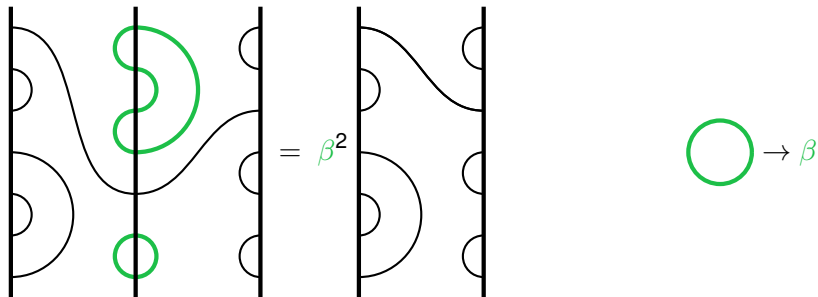
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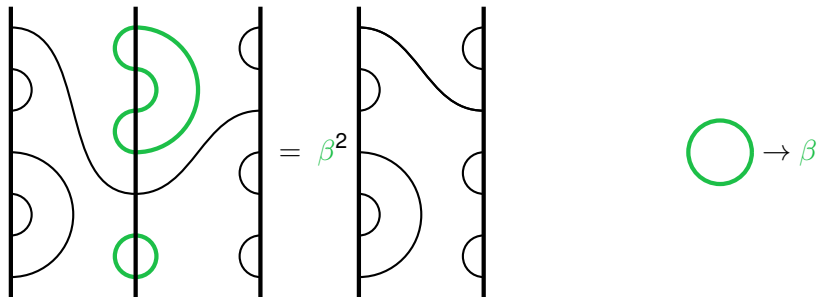
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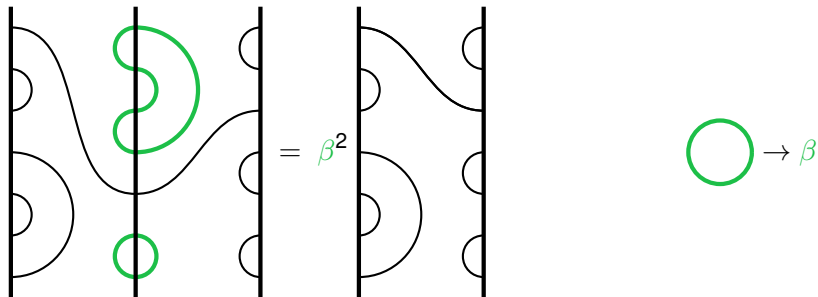
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- Dimensions given by Catalan numbers

One-boundary TL algebra $\mathrm{TL}_n^1(\beta; \alpha_1, \alpha_2)$

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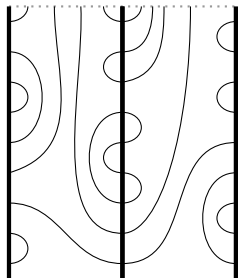
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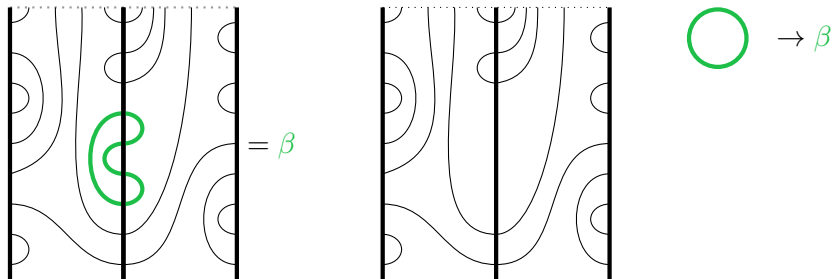
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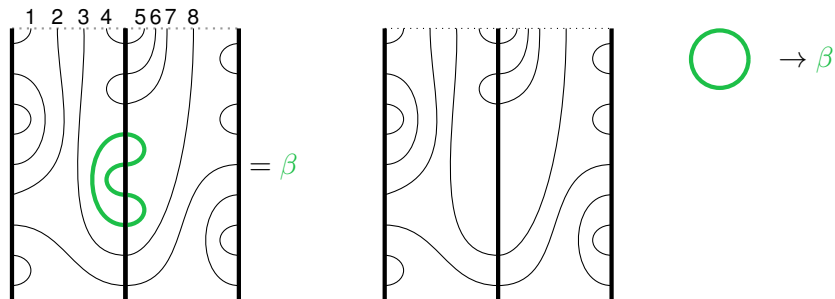
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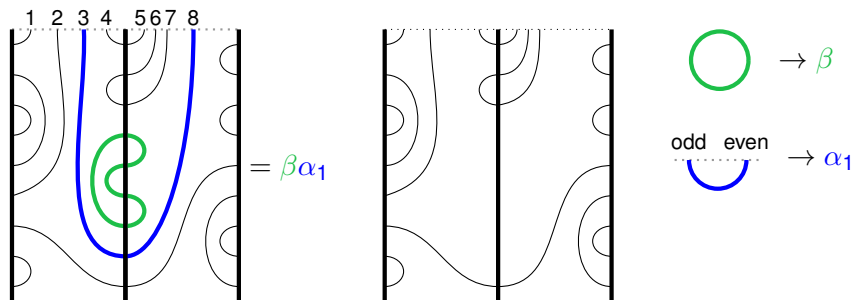
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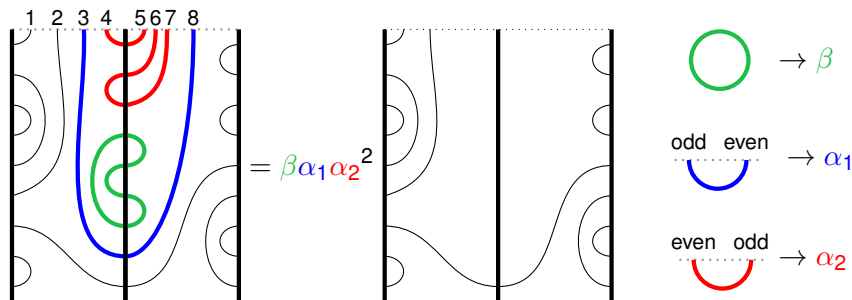
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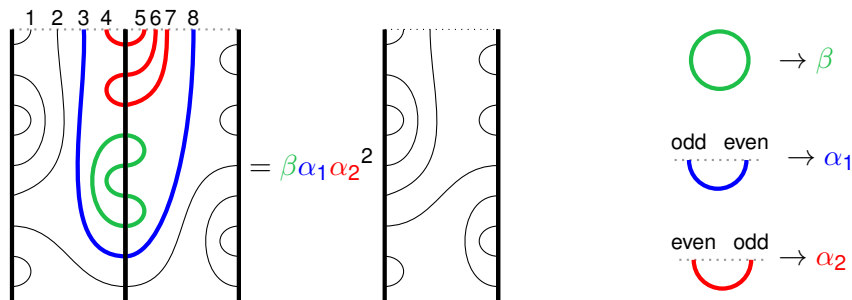
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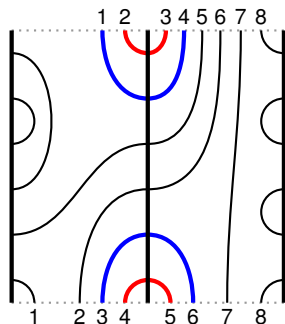
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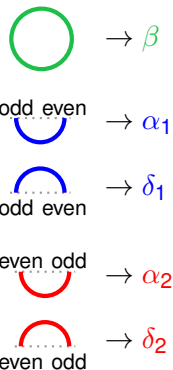
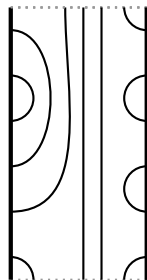
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$$= \alpha_1 \alpha_2 \delta_1 \delta_2$$



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- Finite-dimensional

The naive approach

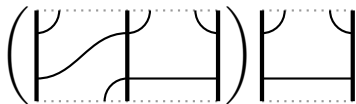
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The naive approach

- Allow odd number of strings at each boundary
- TL_n^2 multiplication rules

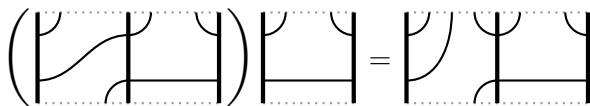
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The diagram illustrates a multiplication rule in the algebra TL_n^2 . It shows three diagrams connected by equals signs. The first diagram is a product of two elements: a pair of pants (left) and a cap (right). The second diagram is a pair of pants with a horizontal line connecting its two bottom legs. The third diagram is a pair of pants with a horizontal line connecting its two bottom legs, and a blue arc connecting the top legs, labeled with indices 1, 2, 3, 4, and 5. The fourth diagram is a pair of pants with a horizontal line connecting its two bottom legs, and a blue arc connecting the top legs, labeled with the index α_1 .

The naive approach

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The diagram shows an equation in the Temperley-Lieb algebra TL_n^2 . On the left, a pair of pants with a crossing on the left side is multiplied by a pair of pants. The result is a pair of pants with a blue arc connecting the top two strings, labeled with indices 1 through 5. This is equal to α_1 times a pair of pants.

The diagram shows a pair of pants with a crossing on the left side multiplied by a pair of pants.

The naive approach

- Allow odd number of strings at each boundary
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$(\text{Diagram 1}) (\text{Diagram 2}) = \text{Diagram 3} = \alpha_1 (\text{Diagram 4})$

$(\text{Diagram 1}) (\text{Diagram 2}) = \text{Diagram 3}$

The naive approach

- Allow odd number of strings at each boundary
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A diagrammatic equation in the algebra TL_n^2 . On the left, a pair of parentheses encloses a diagram with two vertical strings that cross each other. To the right of this is another diagram with two vertical strings that do not cross. An equals sign follows. The next diagram has five vertical strings labeled 1 through 5 at the top. A blue arc connects strings 3 and 4. This is followed by another equals sign and the coefficient α_1 , and finally a diagram with two vertical strings that do not cross.

A diagrammatic equation in the algebra TL_n^2 . On the left, a diagram with two vertical strings that cross is followed by a pair of parentheses enclosing a diagram with four vertical strings labeled 1 through 4 at the top. A red arc connects strings 2 and 3. This is followed by an equals sign and the coefficient α_2 , and finally a diagram with two vertical strings that cross.

The naive approach

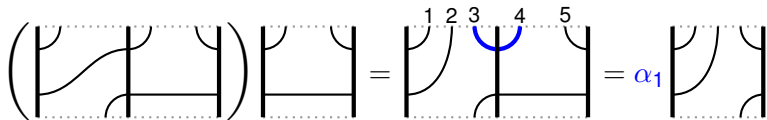
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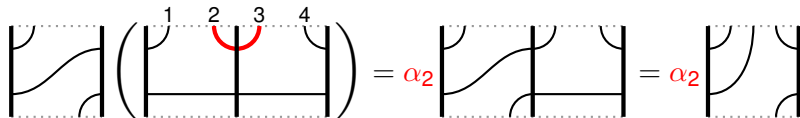
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The naive approach

- Allow odd number of strings at each boundary
- TL_n^2 multiplication rules



A diagrammatic equation in the theory TL_n^2 . On the left, a pair of parentheses encloses a diagram with two vertical strings and a wavy line connecting them. This is multiplied by a diagram with two vertical strings and a horizontal line connecting them. The result is a diagram with five vertical strings labeled 1 through 5 at the top. A blue arc connects strings 3 and 4. This is equal to α_1 times a diagram with two vertical strings and a wavy line connecting them.



A diagrammatic equation in the theory TL_n^2 . On the left, a diagram with two vertical strings and a wavy line connecting them is multiplied by a pair of parentheses enclosing a diagram with four vertical strings labeled 1 through 4 at the top. A red arc connects strings 2 and 3. The result is α_2 times a diagram with two vertical strings and a wavy line connecting them, which is equal to α_2 times a diagram with two vertical strings and a wavy line connecting them.

- Associativity implies $\alpha_1 = \alpha_2$

QUESTION 2:

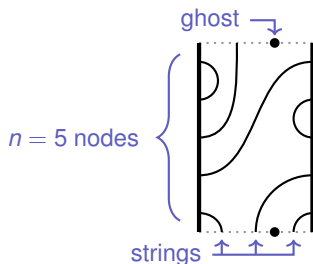
What is the ghost algebra?

The ghost algebra Gh_n^2

Basis of two-boundary n -diagrams with **ghosts**: dots on boundaries that keep track of string parity

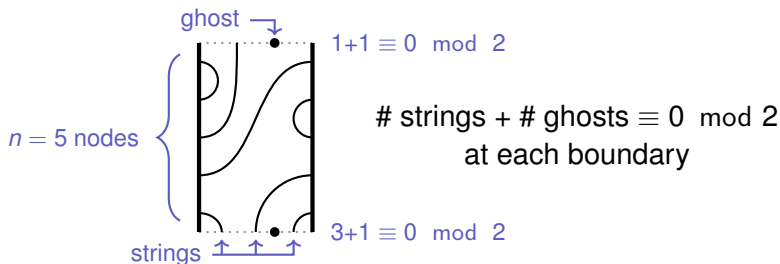
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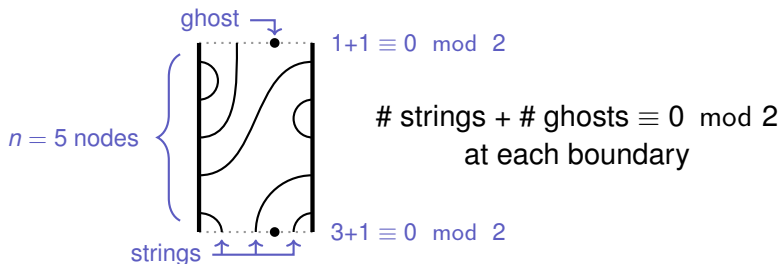
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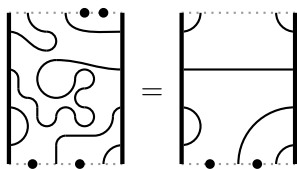


The ghost algebra Gh_n^2

Basis of two-boundary n -diagrams with **ghosts**: dots on boundaries that keep track of string parity



Equality: string isotopy; same # ghosts in each boundary region mod 2

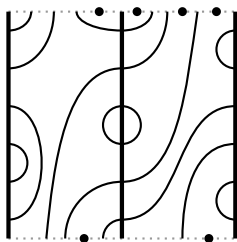


Ghost algebra multiplication

Boundary-to-boundary strings leave a ghost behind at each endpoint

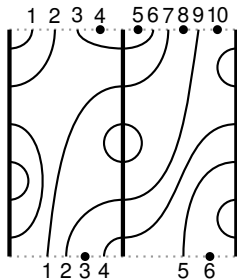
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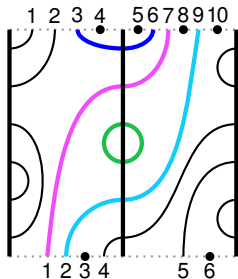
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$$\begin{array}{c} \text{odd} \quad \text{even} \\ \text{---} \quad \text{---} \\ \text{---} \end{array} \rightarrow \alpha_1$$

$$\begin{array}{c} \text{even} \quad \text{odd} \\ \text{---} \quad \text{---} \\ \text{---} \end{array} \rightarrow \alpha_2$$

$$\begin{array}{c} \text{odd} \quad \text{odd} \\ \text{---} \quad \text{---} \\ \text{---} \end{array}, \begin{array}{c} \text{even} \quad \text{even} \\ \text{---} \quad \text{---} \\ \text{---} \end{array} \rightarrow \alpha_3$$

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \rightarrow \delta_1$$

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$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}, \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \rightarrow \delta_3$$

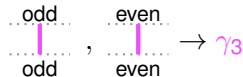
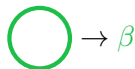
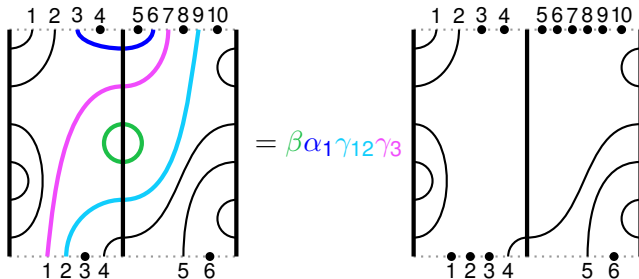
$$\bigcirc \rightarrow \beta$$

$$\begin{array}{c} \text{odd} \\ | \\ \text{---} \\ \text{even} \end{array}, \begin{array}{c} \text{even} \\ | \\ \text{---} \\ \text{odd} \end{array} \rightarrow \gamma_{12}$$

$$\begin{array}{c} \text{odd} \\ | \\ \text{---} \\ \text{odd} \end{array}, \begin{array}{c} \text{even} \\ | \\ \text{---} \\ \text{even} \end{array} \rightarrow \gamma_3$$

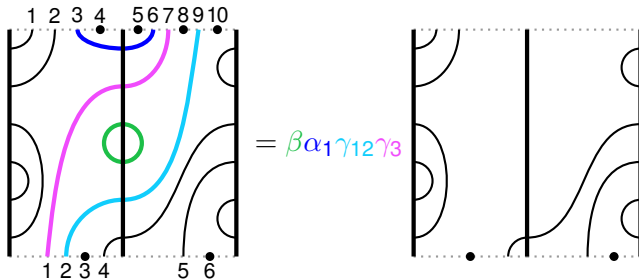
Ghost algebra multiplication

Boundary-to-boundary strings leave a ghost behind at each endpoint



Ghost algebra multiplication

Boundary-to-boundary strings leave a ghost behind at each endpoint



odd even $\rightarrow \alpha_1$

even odd $\rightarrow \alpha_2$

odd odd, even even $\rightarrow \alpha_3$

odd even $\rightarrow \delta_1$

even odd $\rightarrow \delta_2$

odd odd, even even $\rightarrow \delta_3$

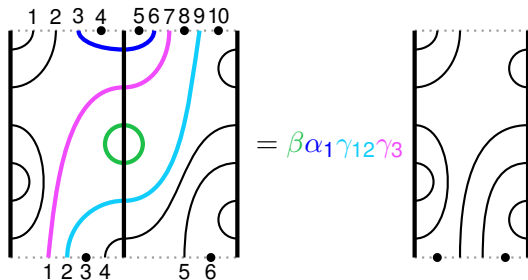
$\rightarrow \beta$

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odd, even $\rightarrow \gamma_3$

Ghost algebra multiplication

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Why could TL_n^2 require an even number of strings at each boundary?

Evenness condition in TL_n^2

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In TL_n^1 ,

$$(\# \text{ strings on bdy}) = (\text{total } \# \text{ string endpoints}) - (\# \text{ nodes})$$

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What if some nodes didn't have any strings connected to them?

QUESTION 3:

What is the dilute ghost algebra?

- 1990s Nienhuis, Warnaar, Seaton, Grimm, Pearce, Zhou, Roche, Batchelor, Yung — loop models, **dilute TL algebra** $dTL_n(\beta)$

Dilute TL algebra

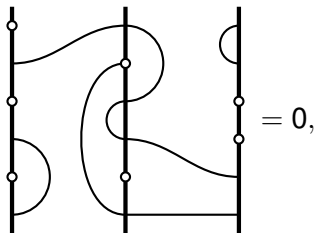
- 1990s Nienhuis, Warnaar, Seaton, Grimm, Pearce, Zhou, Roche, Batchelor, Yung — loop models, **dilute TL algebra** $dTL_n(\beta)$
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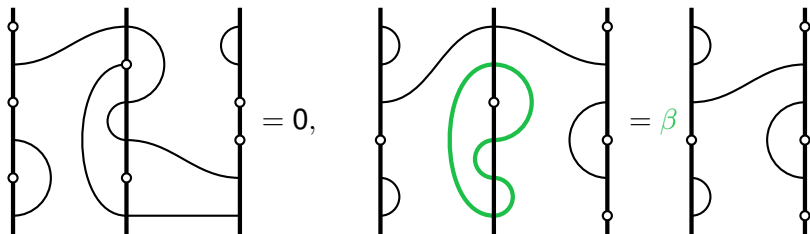
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Boundary dilute algebras

What about one- or two-boundary dilute TL algebras?

Boundary dilute algebras

What about one- or two-boundary dilute TL algebras?

A diagrammatic equation involving two boundary dilute TL algebras. The left side shows the product of two diagrams: the first is a box with a wavy line on the left and a horizontal line at the bottom, and the second is a box with a horizontal line at the bottom. The right side shows a box with a blue arc connecting strands 3 and 4, and a coefficient α_1 to its right.

$$\left(\begin{array}{c} \text{[Diagram 1]} \\ \text{[Diagram 2]} \end{array} \right) = \begin{array}{c} \text{[Diagram 3]} \\ \text{[Diagram 4]} \end{array} = \alpha_1 \begin{array}{c} \text{[Diagram 5]} \\ \text{[Diagram 6]} \end{array}$$

A diagrammatic equation involving two boundary dilute TL algebras. The left side shows a box with a wavy line on the left and a horizontal line at the bottom, multiplied by a box with a red arc connecting strands 2 and 3. The right side shows a box with a wavy line on the left and a horizontal line at the bottom, multiplied by a coefficient α_2 .

$$\begin{array}{c} \text{[Diagram 1]} \\ \text{[Diagram 2]} \end{array} \left(\begin{array}{c} \text{[Diagram 3]} \\ \text{[Diagram 4]} \end{array} \right) = \alpha_2 \begin{array}{c} \text{[Diagram 5]} \\ \text{[Diagram 6]} \end{array} = \alpha_2 \begin{array}{c} \text{[Diagram 7]} \\ \text{[Diagram 8]} \end{array}$$

Boundary dilute algebras

What about one- or two-boundary dilute TL algebras?

A diagrammatic equation in the boundary dilute TL algebra. On the left, a pair of pants with a wavy line on the left leg and a dot on the right leg is enclosed in large parentheses, followed by a pair of pants with a dot on the left leg. This is equal to a pair of pants with a dot on the left leg, followed by a blue arc connecting the top two legs labeled 2 and 3, and a pair of pants with a dot on the left leg. The right side of the equation is equal to α_1 times a pair of pants with a wavy line on the left leg and a dot on the right leg, followed by a pair of pants with a dot on the left leg. A dotted horizontal line is drawn above the diagrams, with labels 1, 2, 3, 4, and 5 above it.

A diagrammatic equation in the boundary dilute TL algebra. On the left, a pair of pants with a wavy line on the left leg and a dot on the right leg is enclosed in large parentheses, followed by a pair of pants with a dot on the left leg, a red arc connecting the top two legs labeled 2 and 3, and a pair of pants with a dot on the left leg. This is equal to α_2 times a pair of pants with a wavy line on the left leg and a dot on the right leg, followed by a pair of pants with a dot on the left leg. This is equal to α_2 times a pair of pants with a wavy line on the left leg and a dot on the right leg, followed by a pair of pants with a dot on the left leg. A dotted horizontal line is drawn above the diagrams, with labels 1, 2, 3, and 4 above it.

Boundary associativity problem occurs even in one-boundary case.

Boundary dilute algebras

What about one- or two-boundary dilute TL algebras?

A diagrammatic equation involving strands and a dotted horizontal line. On the left, a pair of strands with a crossing is enclosed in large parentheses, followed by a single strand with a dot below it. This is equal to a strand with a dot below it, followed by a strand with a dot below it, then a blue arc connecting strands 3 and 4, and finally a strand with a dot below it. This is equal to α_1 times a pair of strands with a crossing, followed by a single strand with a dot below it. The strands are numbered 1 through 5 at the top.

A diagrammatic equation involving strands and a dotted horizontal line. On the left, a pair of strands with a crossing is enclosed in large parentheses, followed by a single strand with a dot below it. This is equal to α_2 times a pair of strands with a crossing, followed by a single strand with a dot below it. This is equal to α_2 times a pair of strands with a crossing, followed by a single strand with a dot below it. The strands are numbered 1 through 4 at the top.

Boundary associativity problem occurs even in one-boundary case.

No algebras in literature have $dTL_n(\beta)$ and $TL_n^1(\beta; \alpha_1, \alpha_2)$ as subalgebras with $\alpha_1 \neq \alpha_2$.

Dilute ghost algebra dGh_n^2

Dilute ghost algebra dGh_n^2 allows n -diagrams with empty nodes

Dilute ghost algebra dGh_n^2

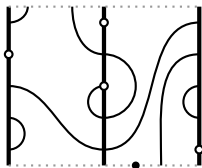
Dilute ghost algebra dGh_n^2 allows n -diagrams with empty nodes

String to empty node gives 0:

Dilute ghost algebra dGh_n^2

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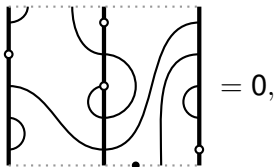
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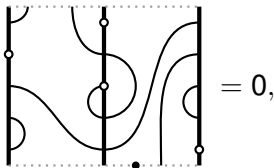
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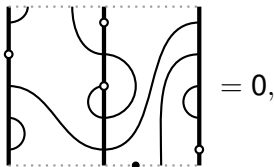


Same parameters as Gh_n^2 :

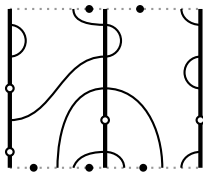
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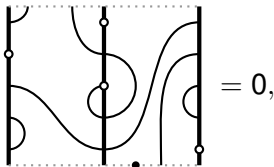
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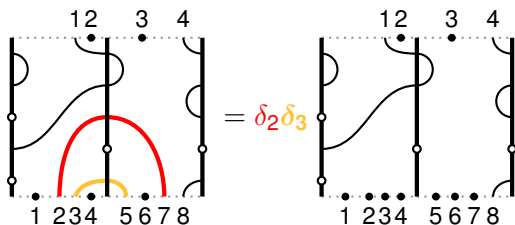
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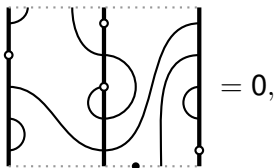
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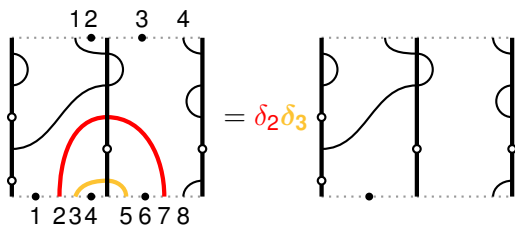
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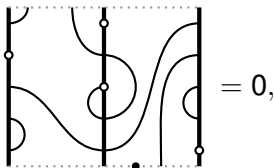
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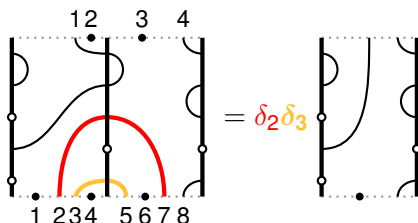
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$$\text{Gh}_n^2 \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta & \gamma_{12} & \gamma_3 \\ \delta_1 & \delta_2 & \delta_3 \end{pmatrix}$$

$$\mathrm{Gh}_n^1(\beta; \alpha_1, \alpha_2, \alpha_3) \hookrightarrow \mathrm{Gh}_n^2 \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta & \gamma_{12} & \gamma_3 \\ \delta_1 & \delta_2 & \delta_3 \end{pmatrix}$$

Algebra inclusions

$$\begin{array}{ccc} \mathrm{TL}_n^1(\beta; \alpha_1, \alpha_2) & & \\ \downarrow & & \\ \mathrm{Gh}_n^1(\beta; \alpha_1, \alpha_2, \alpha_3) & \hookrightarrow & \mathrm{Gh}_n^2 \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta & \gamma_{12} & \gamma_3 \\ \delta_1 & \delta_2 & \delta_3 \end{pmatrix} \end{array}$$

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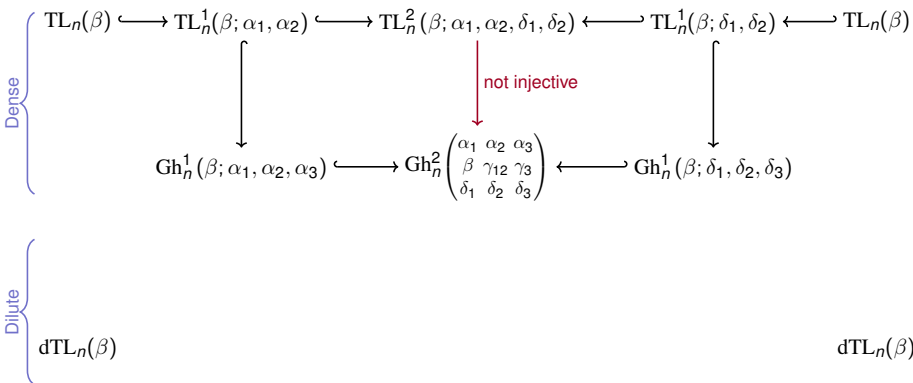
$$\begin{array}{ccccccc} \mathrm{TL}_n(\beta) & \longleftrightarrow & \mathrm{TL}_n^1(\beta; \alpha_1, \alpha_2) & \longleftrightarrow & \mathrm{TL}_n^2(\beta; \alpha_1, \alpha_2, \delta_1, \delta_2) & \longleftrightarrow & \mathrm{TL}_n^1(\beta; \delta_1, \delta_2) & \longleftrightarrow & \mathrm{TL}_n(\beta) \\ & & \downarrow & & \downarrow \text{not injective} & & \downarrow & & \\ \mathrm{Gh}_n^1(\beta; \alpha_1, \alpha_2, \alpha_3) & \longleftrightarrow & \mathrm{Gh}_n^2 \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta & \gamma_{12} & \gamma_3 \\ \delta_1 & \delta_2 & \delta_3 \end{pmatrix} & \longleftrightarrow & \mathrm{Gh}_n^1(\beta; \delta_1, \delta_2, \delta_3) & & & & \end{array}$$

Algebra inclusions

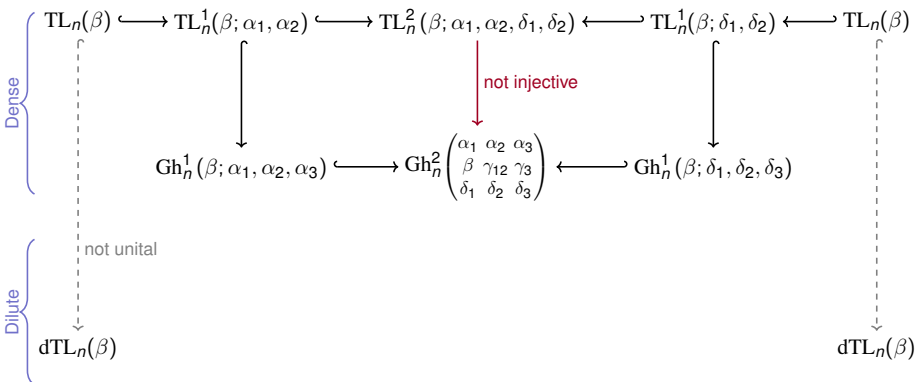
Dense

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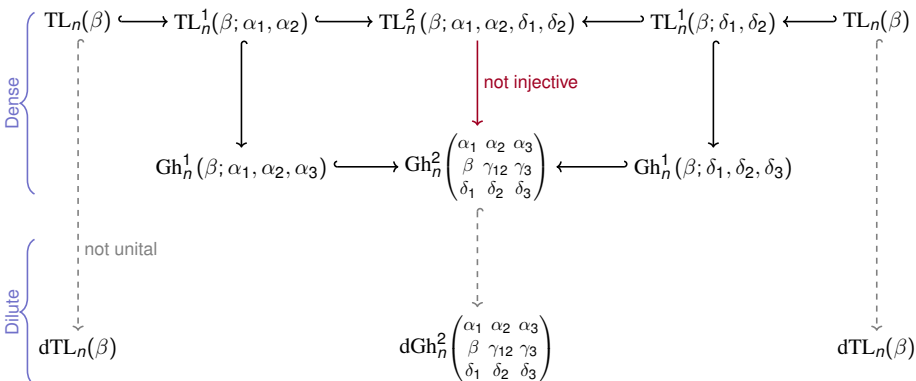
Algebra inclusions



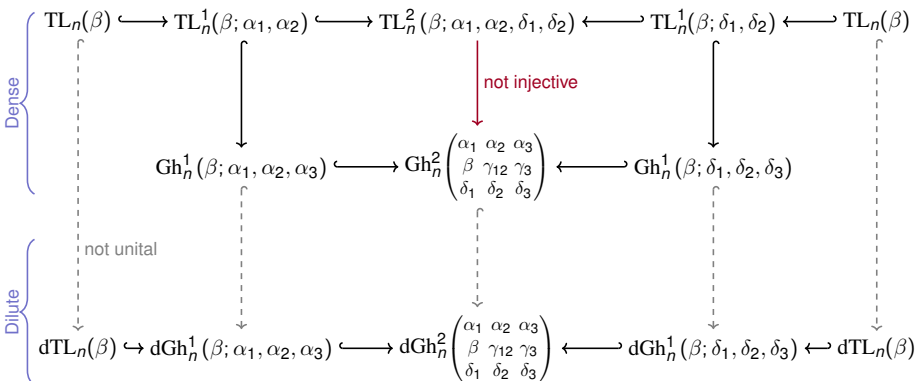
Algebra inclusions



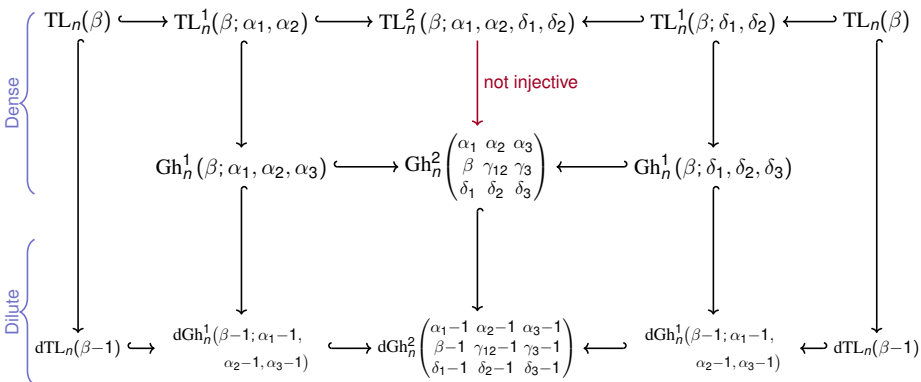
Algebra inclusions



Algebra inclusions



Algebra inclusions

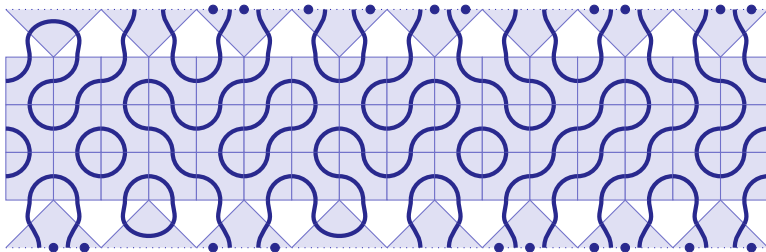


QUESTION 4:

What can we do with these algebras?

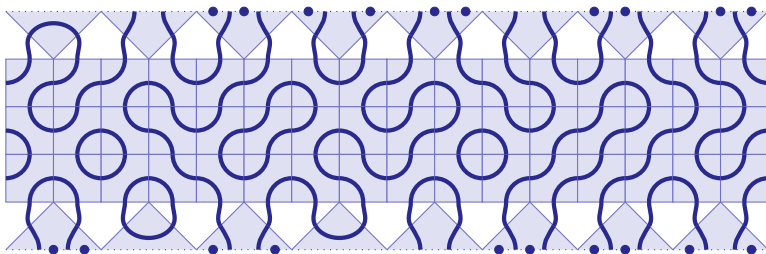
Loop model

Build a lattice out of **bulk squares** and **boundary triangles**



Loop model

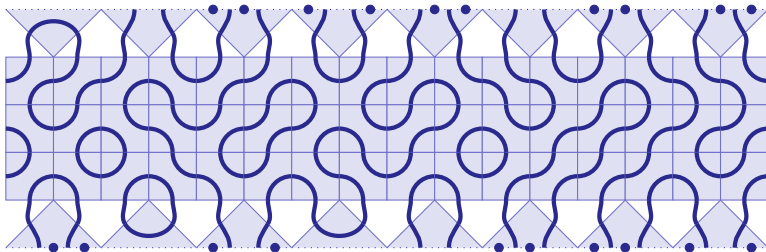
Build a lattice out of **bulk squares** and **boundary triangles**



Construct Hamiltonians using solutions to **Yang-Baxter equation (YBE)** and **boundary Yang-Baxter equation (BYBE)**.

Loop model

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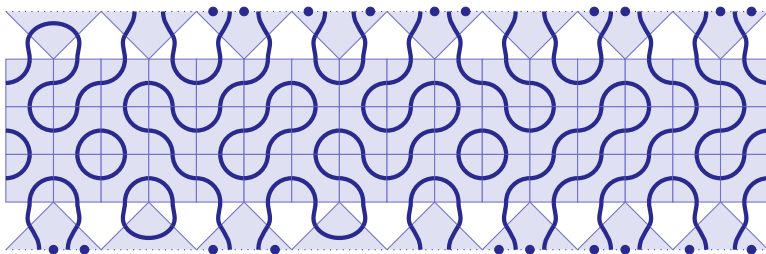


Construct Hamiltonians using solutions to **Yang-Baxter equation (YBE)** and **boundary Yang-Baxter equation (BYBE)**.

YBE:

Loop model

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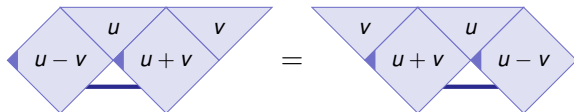
Construct Hamiltonians using solutions to **Yang-Baxter equation (YBE)** and **boundary Yang-Baxter equation (BYBE)**.

YBE:

$$\diamond u := \frac{\sin(\lambda - u)}{\sin(\lambda)} \diamond \text{ (wavy lines) } + \frac{\sin(u)}{\sin(\lambda)} \diamond \text{ (curved lines) }, \quad \beta = 2 \cos \lambda$$

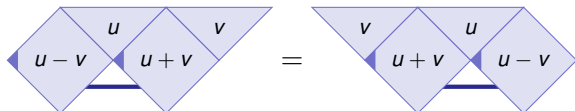
Boundary Yang-Baxter equation

The BYBE involves **boundary face operators**



Boundary Yang-Baxter equation

The BYBE involves **boundary face operators**

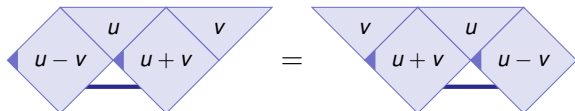


Two solutions: $\rho \in \mathbb{C}$, $\beta = 2 \cos \lambda$, $b : \mathbb{C} \rightarrow \mathbb{C}$,

$$\triangleleft_u = b(u) \left(\frac{F(u)}{\sin(2u)} \triangleleft_u + c_1 \triangleleft_u + c_2 \triangleleft_u + c_3 \triangleleft_u + c_4 \triangleleft_u \right)$$

Boundary Yang-Baxter equation

The BYBE involves **boundary face operators**



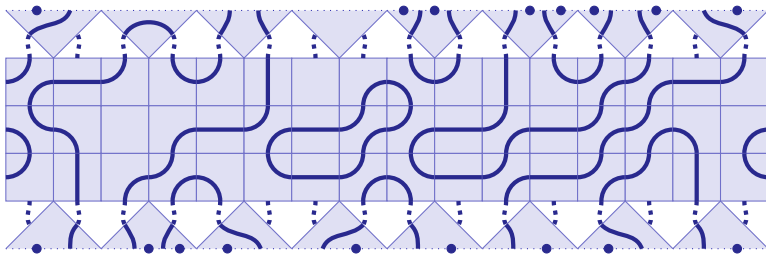
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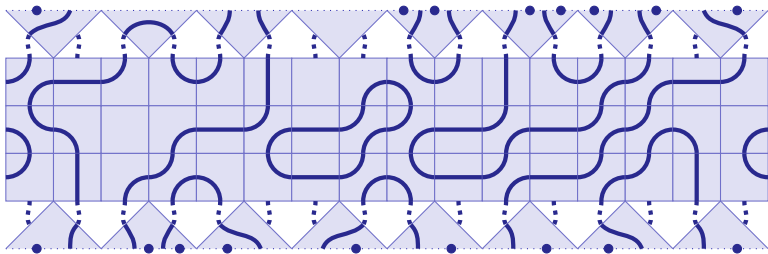
$$\begin{cases} F(u) = \rho + (c_1 \alpha_1 + c_3 \alpha_2 + (c_2 + c_4) \alpha_3) \cos(2u) \\ \quad - (c_3 \alpha_1 + c_1 \alpha_2 + (c_2 + c_4) \alpha_3) \cos(2u - \lambda), \\ c_1 c_3 = c_2 c_4, \end{cases}$$

$$\begin{cases} F(u) = \rho + (2\alpha_3^2 - \alpha_1^2 - \alpha_2^2) \cos(2u) - (\alpha_3^2 - \alpha_1 \alpha_2) \cos(2u - \lambda), \\ c_1 = -\alpha_1, \quad c_2 = c_4 = \alpha_3, \quad c_3 = -\alpha_2. \end{cases}$$

Dilute loop model



Dilute loop model



Batchelor & Yung (1995) — YBE solution ($\beta = -2 \cos(4\phi)$)

$$\begin{aligned}
 \diamond_u &:= w_1(u) \diamond + w_2(u) \left(\diamond_{\text{arc}} + \diamond_{\text{arc}} \right) + w_3(u) \left(\diamond_{\text{arc}} + \diamond_{\text{arc}} \right) \\
 &+ w_4(u) \left(\diamond_{\text{diag}} + \diamond_{\text{diag}} \right) + w_5(u) \diamond_{\text{diag}} + w_6(u) \diamond_{\text{diag}}
 \end{aligned}$$

$$w_1(u) = \sin(2\phi) \sin(3\phi) + \sin u \sin(3\phi - u)$$

$$w_2(u) = \sin(2\phi) \sin(3\phi - u)$$

$$w_3(u) = \sin(2\phi) \sin u$$

$$w_4(u) = \sin u \sin(3\phi - u)$$

$$w_5(u) = \sin(2\phi - u) \sin(3\phi - u)$$

$$w_6(u) = -\sin u \sin(\phi - u)$$

Dilute ghost algebra BYBE solution

$$\begin{aligned} \triangleleft u &= a(u) \triangleleft \text{arc} + b_1(u) \triangleleft \text{empty} + b_2(u) \triangleleft \text{loop} + b_3(u) \triangleleft \text{dot} + b_4(u) \triangleleft \text{dot} \\ &+ b_5(u) \triangleleft \text{dot} + b_6(u) \triangleleft \text{dot} + b_7(u) \triangleleft \text{dot} + b_8(u) \triangleleft \text{dot} + b_9(u) \triangleleft \text{dot} \end{aligned}$$

Dilute ghost algebra BYBE solution

$$\begin{aligned}
 \triangle u &= a(u) \triangle + b_1(u) \triangle + b_2(u) \triangle + b_3(u) \triangle + b_4(u) \triangle \\
 &+ b_5(u) \triangle + b_6(u) \triangle + b_7(u) \triangle + b_8(u) \triangle + b_9(u) \triangle
 \end{aligned}$$

$$a(u) = \frac{-\hbar(u) \sin\left(u - \frac{3\phi}{2}\right)}{\sin(2u) \sin(2\phi)} \left(2\rho^2 \cos(\phi) - \kappa^2 \left((\rho^2 + \nu^2) \alpha_2 + \mu\nu(\alpha_1 + \alpha_2) \right) \cos^2\left(u - \frac{\phi}{2}\right) \right)$$

$$b_1(u) = \frac{\hbar(u)}{\sin(2u) \sin(2\phi)} \left(2\rho^2 \sin\left(u + \frac{3\phi}{2}\right) \cos(\phi) - \kappa^2 \left((\rho^2 + \nu^2) \alpha_2 + \mu\nu(\alpha_1 + \alpha_2) \right) \sin\left(u + \frac{\phi}{2}\right) \sin\left(u - \frac{\phi}{2}\right) \sin\left(u - \frac{3\phi}{2}\right) \right)$$

$$b_2(u) = b_3(u) = \nu\kappa\rho\hbar(u), \quad b_4(u) = b_5(u) = \mu\kappa\rho\hbar(u)$$

$$b_6(u) = b_7(u) = \mu\nu\kappa^2\hbar(u) \sin\left(u - \frac{\phi}{2}\right)$$

$$b_8(u) = \nu^2\kappa^2\hbar(u) \sin\left(u - \frac{\phi}{2}\right), \quad b_9(u) = \mu^2\kappa^2\hbar(u) \sin\left(u - \frac{\phi}{2}\right)$$

$$a(u) = \frac{\hbar(u) \cos\left(u - \frac{3\phi}{2}\right)}{\sin(2u) \sin(2\phi)} \left(2\rho^2 \cos(\phi) - \kappa^2 \left((\rho^2 + \nu^2) \alpha_2 - \mu\nu(\alpha_1 + \alpha_2) \right) \cos^2\left(u - \frac{\phi}{2}\right) \right)$$

$$b_1(u) = \frac{\hbar(u)}{\sin(2u) \sin(2\phi)} \left(2\rho^2 \cos\left(u + \frac{3\phi}{2}\right) \cos(\phi) - \kappa^2 \left((\rho^2 + \nu^2) \alpha_2 - \mu\nu(\alpha_1 + \alpha_2) \right) \cos\left(u + \frac{\phi}{2}\right) \cos\left(u - \frac{\phi}{2}\right) \cos\left(u - \frac{3\phi}{2}\right) \right)$$

$$b_2(u) = \nu\kappa\rho\hbar(u), \quad b_3(u) = -\tau\kappa\rho\hbar(u), \quad b_4(u) = \tau\kappa\rho\hbar(u), \quad b_5(u) = -\nu\kappa\rho\hbar(u)$$

$$b_6(u) = b_7(u) = -\mu\nu\kappa^2\hbar(u) \cos\left(u - \frac{\phi}{2}\right)$$

$$b_8(u) = \nu^2\kappa^2\hbar(u) \cos\left(u - \frac{\phi}{2}\right), \quad b_9(u) = \mu^2\kappa^2\hbar(u) \cos\left(u - \frac{\phi}{2}\right)$$

$$a(u) = \frac{-\hbar(u)}{2 \sin(2u)} \frac{(\alpha_1^2 + \alpha_2^2 - 2\alpha_1^2) (\cos(u) - \kappa \sin(u)) (\cos(u - \phi) - \kappa \sin(u - \phi))}{\left(\kappa \sin\left(\frac{u}{2}\right) - \cos\left(\frac{u - \phi}{2}\right) \right) \left(\kappa \cos\left(\frac{u}{2}\right) + \sin\left(\frac{u - \phi}{2}\right) \right)}$$

$$b_1(u) = \frac{-\hbar(u)}{2 \sin(2u)} \frac{(\alpha_1^2 + \alpha_2^2 - 2\alpha_1^2) (\cos(u) + \kappa \sin(u)) (\cos(u - \phi) - \kappa \sin(u - \phi))}{\left(\kappa \sin\left(\frac{u}{2}\right) - \cos\left(\frac{u - \phi}{2}\right) \right) \left(\kappa \cos\left(\frac{u}{2}\right) + \sin\left(\frac{u - \phi}{2}\right) \right)}$$

$$b_2(u) = b_3(u) = b_4(u) = b_5(u) = 0$$

$$b_6(u) = -\alpha_1 \hbar(u), \quad b_7(u) = b_8(u) = \alpha_2 \hbar(u), \quad b_9(u) = -\alpha_2 \hbar(u)$$

$$(\kappa^2 - 1) \left((\alpha_1 \alpha_2 - \alpha_2^2) \sin(2\phi) - (\alpha_1^2 + \alpha_2^2 - 2\alpha_2^2) \sin(\phi) \right) = 2\kappa \left((\alpha_1 \alpha_2 - \alpha_2^2) \cos(2\phi) + (\alpha_1^2 + \alpha_2^2 - 2\alpha_2^2) \cos(\phi) \right)$$

$$a(u) = \frac{-\hbar(u)}{\sin(2u)} (\alpha_1 - \alpha_2) \left(\mu\nu(\alpha_1 + \alpha_2) + (\rho^2 + \nu^2) \alpha_2 \right) (\tau \cos(u) - \tau \sin(u)) (\tau \cos(u - \phi) - \tau \sin(u - \phi))$$

$$b_1(u) = \frac{-\hbar(u)}{\sin(2u)} (\alpha_1 - \alpha_2) \left(\mu\nu(\alpha_1 + \alpha_2) + (\rho^2 + \nu^2) \alpha_2 \right) (\tau \cos(u) + \tau \sin(u)) (\tau \cos(u - \phi) - \tau \sin(u - \phi))$$

$$b_2(u) = b_3(u) = b_4(u) = b_5(u) = 0$$

$$b_6(u) = \mu\hbar(u) (\nu\alpha_2 + \mu\alpha_2) (2\tau \tau \cos(\phi) - (\rho^2 - \tau^2) \sin(\phi)) + (\nu\alpha_1 + \mu\alpha_2) (2\tau \tau \cos(2\phi) + (\rho^2 - \tau^2) \sin(2\phi))$$

$$b_7(u) = \nu\hbar(u) (\nu\alpha_2 + \mu\alpha_2) (2\tau \tau \cos(\phi) - (\rho^2 - \tau^2) \sin(\phi)) + (\nu\alpha_1 + \mu\alpha_2) (2\tau \tau \cos(2\phi) + (\rho^2 - \tau^2) \sin(2\phi))$$

$$b_8(u) = -\nu\hbar(u) ((\mu\alpha_1 + \nu\alpha_2) (2\tau \tau \cos(\phi) - (\rho^2 - \tau^2) \sin(\phi)) + (\mu\alpha_2 + \nu\alpha_2) (2\tau \tau \cos(2\phi) + (\rho^2 - \tau^2) \sin(2\phi)))$$

$$b_9(u) = -\mu\hbar(u) ((\mu\alpha_1 + \nu\alpha_2) (2\tau \tau \cos(\phi) - (\rho^2 - \tau^2) \sin(\phi)) + (\mu\alpha_2 + \nu\alpha_2) (2\tau \tau \cos(2\phi) + (\rho^2 - \tau^2) \sin(2\phi)))$$

$$a(u) = \frac{(\alpha_2 \alpha_1 + \alpha_2 \alpha_2 + (\alpha_2 + \alpha_2) \alpha_2) \hbar(u) (\tau \cos(u - \phi) + \tau \sin(u - \phi)) (\tau \cos u + \tau \sin u)}{\sin(2u) (2\tau \tau \cos(2\phi) - (\rho^2 - \tau^2) \sin(2\phi))}$$

$$b_1(u) = \frac{(\alpha_2 \alpha_1 + \alpha_2 \alpha_2 + (\alpha_2 + \alpha_2) \alpha_2) \hbar(u) (\tau \cos(u - \phi) + \tau \sin(u - \phi)) (\tau \cos u - \tau \sin u)}{\sin(2u) (2\tau \tau \cos(2\phi) - (\rho^2 - \tau^2) \sin(2\phi))}$$

$$b_2(u) = b_3(u) = b_4(u) = b_5(u) = 0$$

$$b_6(u) = \alpha_2 \hbar(u), \quad b_7(u) = \alpha_2 \hbar(u), \quad b_8(u) = \alpha_2 \hbar(u), \quad b_9(u) = \alpha_2 \hbar(u)$$

$$\alpha_1 \alpha_2 = \alpha_2 \alpha_2, \quad A_1 = \alpha_1^2 + \alpha_2^2 - 2\alpha_2^2, \quad A_2 = \alpha_1 \alpha_2 - \alpha_1^2$$

$$(\alpha_2 \alpha_1 + \alpha_2 \alpha_2 + (\alpha_2 + \alpha_2) \alpha_2) (2\tau \tau \cos \phi + (\rho^2 - \tau^2) \sin \phi) + (\alpha_2 \alpha_1 + \alpha_2 \alpha_2 + (\alpha_2 + \alpha_2) \alpha_2) (2\tau \tau \cos(2\phi) - (\rho^2 - \tau^2) \sin(2\phi))$$

$$(\alpha_2 \alpha_2 + \alpha_2 \alpha_2) (2\tau \tau \cos \phi + (\rho^2 - \tau^2) \sin \phi) + (\alpha_2 \alpha_1 + \alpha_2 \alpha_2) (2\tau \tau \cos(2\phi) - (\rho^2 - \tau^2) \sin(2\phi))$$

$$(\rho^2 + \tau^2)^2 (2\alpha_2 + A_2 \cos(4\phi)) - (\rho^4 - 2\rho^2 \tau^2 + \tau^4) ((A_1 + A_2) \cos(2\phi) + A_2 \cos(\phi)) - 4\tau \tau (\rho^2 - \tau^2) ((A_1 - A_2) \sin(2\phi) + A_2 \sin(\phi)) = 0$$

What else can we do with these algebras?

- Representation theory — use cellularity

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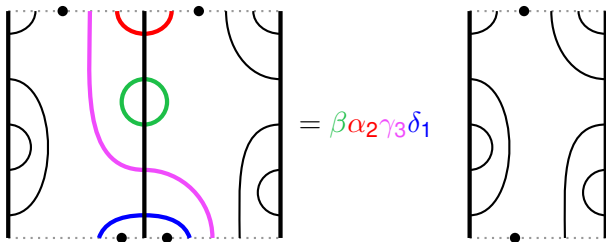
- Representation theory — use cellularity
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What else can we do with these algebras?

- Representation theory — use cellularity
- Physics — Hamiltonians and energy eigenvalues
- Further generalisations

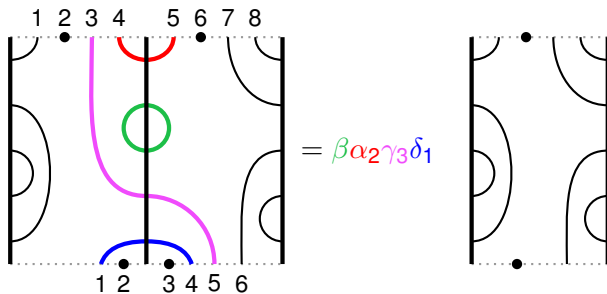
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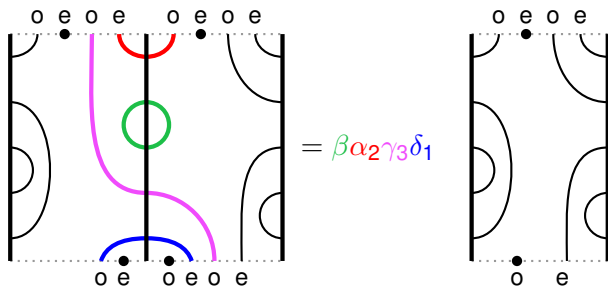
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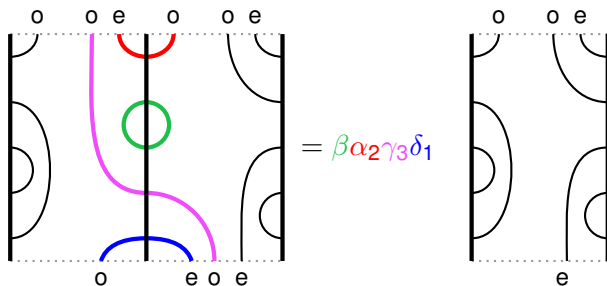
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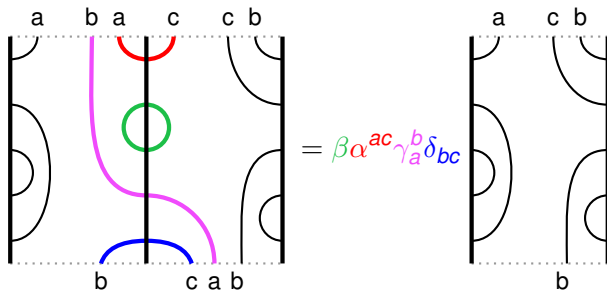
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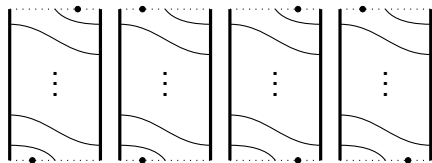
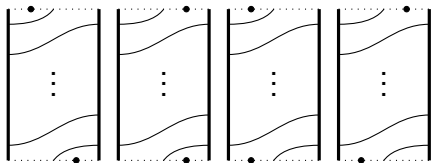
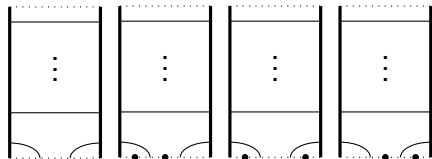
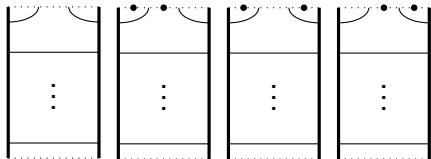
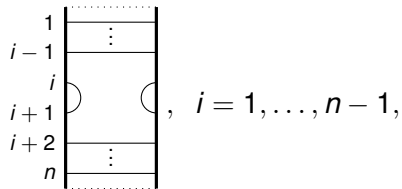
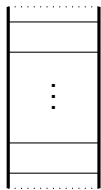


What else can we do with these algebras?

- Representation theory — use cellularity
- Physics — Hamiltonians and energy eigenvalues
- Further generalisations — **label algebra**



Ghost algebra generators



Dimensions

$$\dim \text{Gh}_n^1 = \sum_{d=0}^n \left(\sum_{j=0}^{\lfloor \frac{n-d}{2} \rfloor} 2^{n-2j-d} \left(\binom{n}{j} - \binom{n}{j-1} \right) \right)^2$$

$$\dim \text{Gh}_n^2 = \sum_{d=0}^n \left(\sum_{j=0}^{\lfloor \frac{n-d}{2} \rfloor} 2^{n-2j-d} (n-2j-d+1) \left(\binom{n}{j} - \binom{n}{j-1} \right) \right)^2$$

$$\dim \text{dGh}_n^1 = \sum_{d=0}^n \left(\sum_{v=0}^{n-d} \binom{n}{v} \sum_{j=0}^{\lfloor \frac{n-v-d}{2} \rfloor} 2^{n-v-2j-d} \left(\binom{n-v}{j} - \binom{n-v}{j-1} \right) \right)^2$$

$$\begin{aligned} \dim \text{dGh}_n^2 = \sum_{d=0}^n \left(\sum_{v=0}^{n-d} \binom{n}{v} \sum_{j=0}^{\lfloor \frac{n-v-d}{2} \rfloor} 2^{n-v-2j-d} (n-v-2j-d+1) \right. \\ \left. \times \left(\binom{n-v}{j} - \binom{n-v}{j-1} \right) \right)^2 \end{aligned}$$

Table of dimensions

n	$\dim Gh_n^1$	$\dim Gh_n^2$	$\dim dGh_n^1$	$\dim dGh_n^2$
1	5	17	10	26
2	30	186	117	521
3	185	1,813	1,407	9,355
4	1,150	16,102	17,083	156,947
5	7,170	135,866	208,284	2,514,932
6	44,760	1,099,276	2,544,751	38,968,815
7	279,585	8,639,133	31,125,138	588,475,298
8	1,746,870	66,258,526	380,928,795	8,706,799,523
9	10,916,150	498,701,470	4,663,705,782	126,690,947,758
10	68,219,860	3,693,607,300	57,109,857,519	1,818,028,127,339