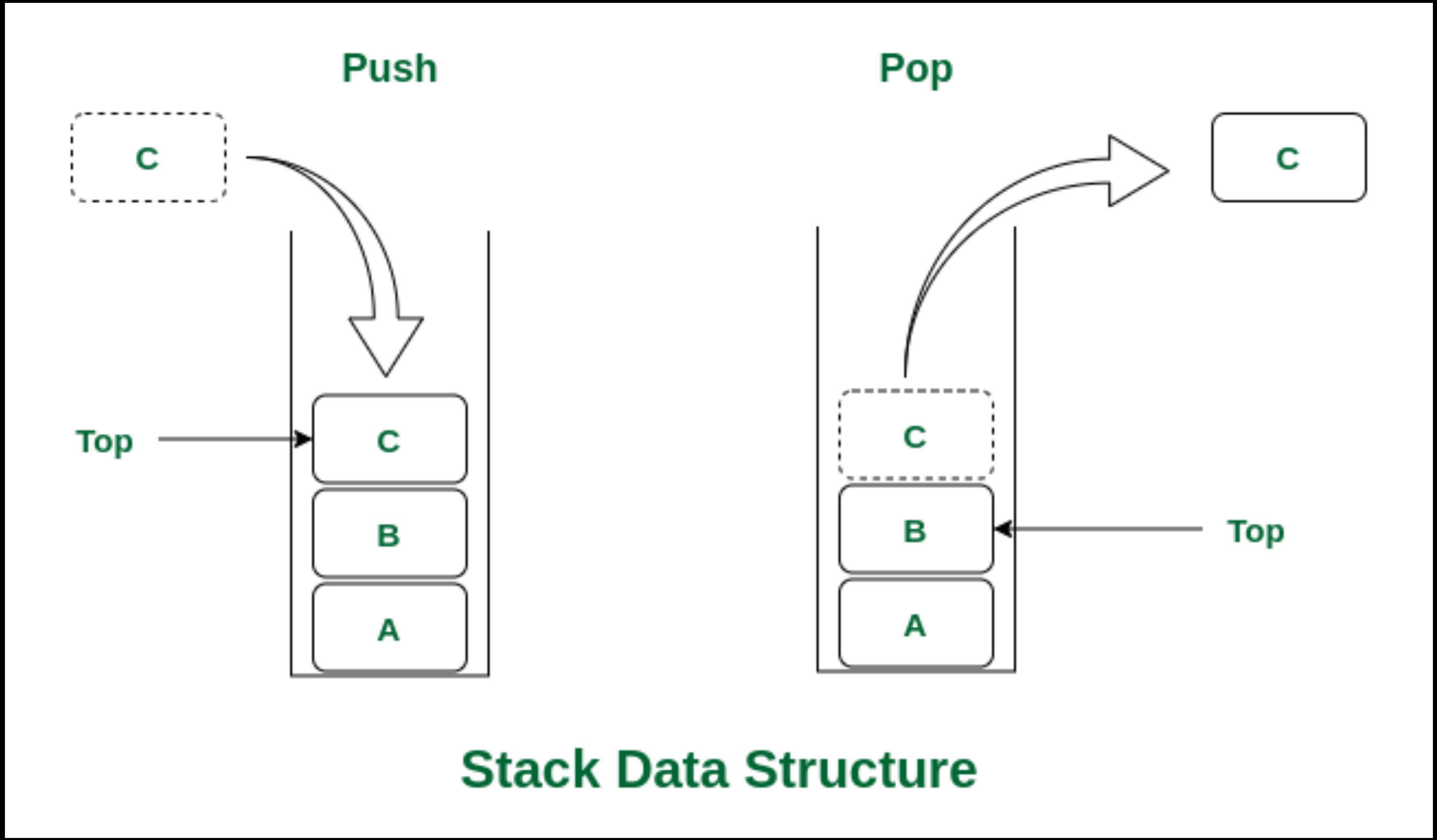
CATALAN NUMBERS — COUNTING IN 🐢

ELLA WANG

2023 MAY

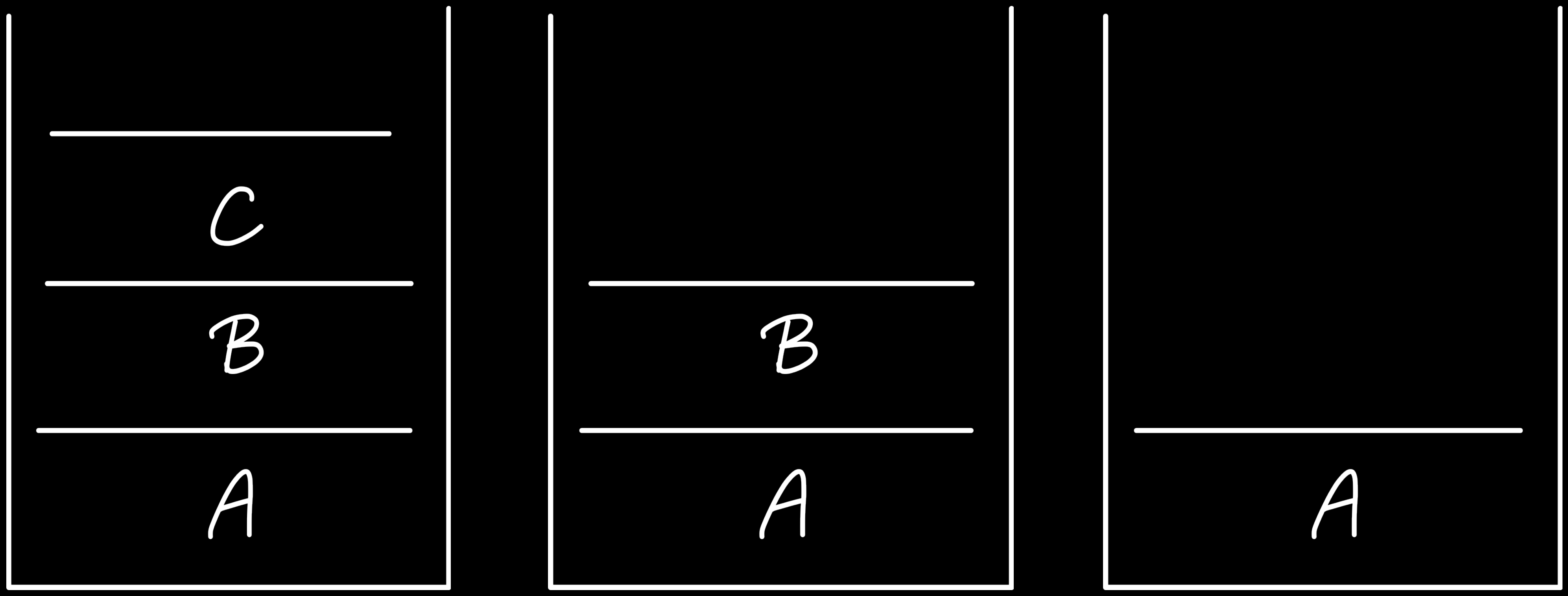
Stack – FILO queue



Stack Data Structure

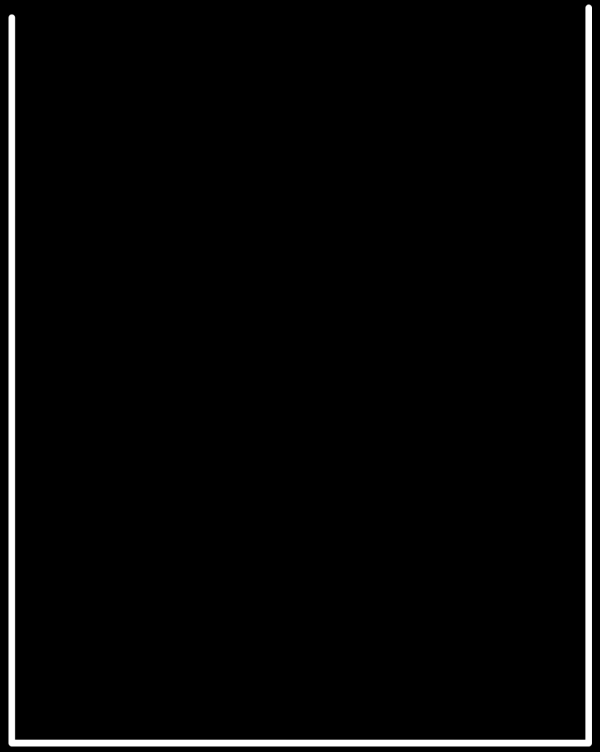
Push order: A B C

Pop order: C B A



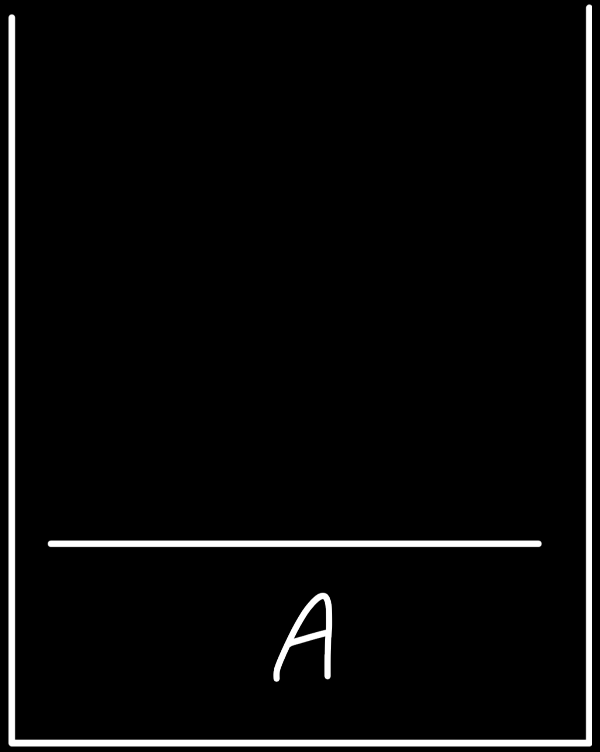
Push order: A B C

Pop order: A B C



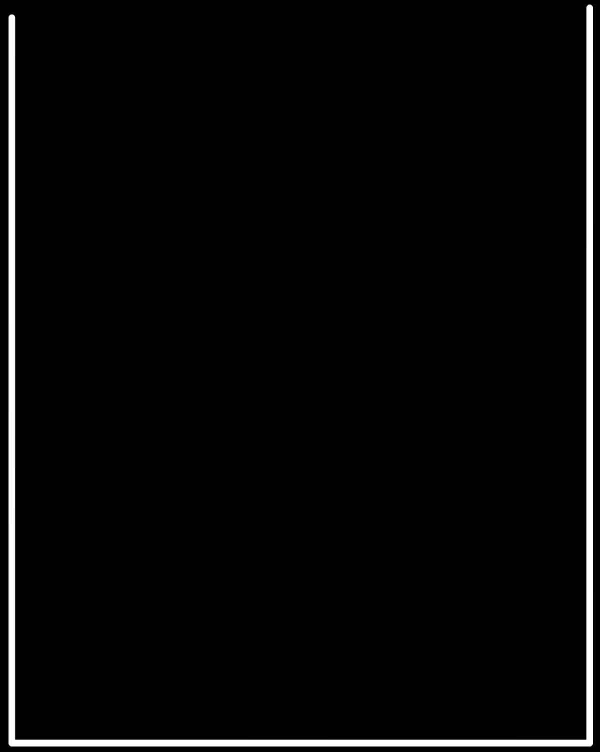
Push order: A B C

Pop order: A B C



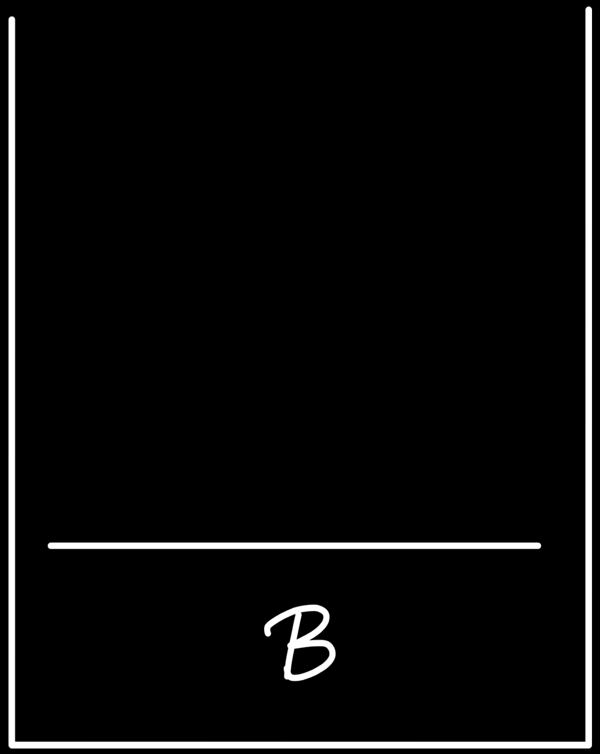
Push order: A B C

Pop order: A B C



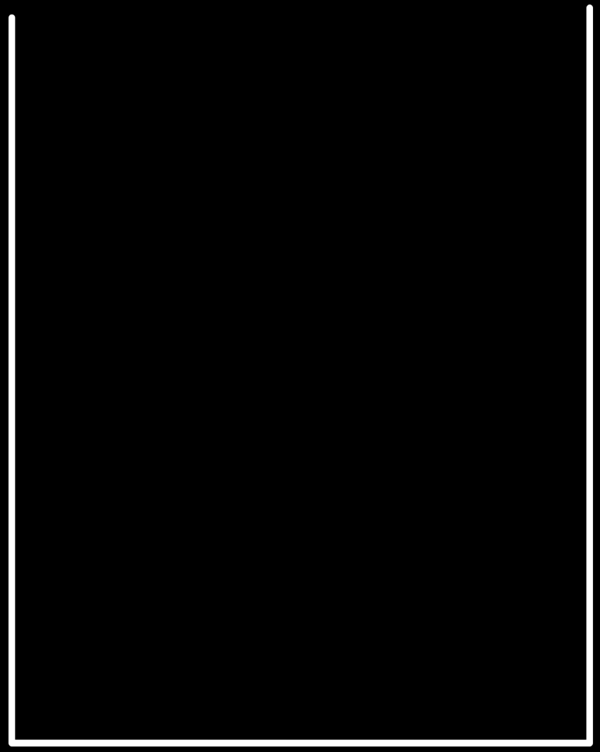
Push order: A B C

Pop order: A B C



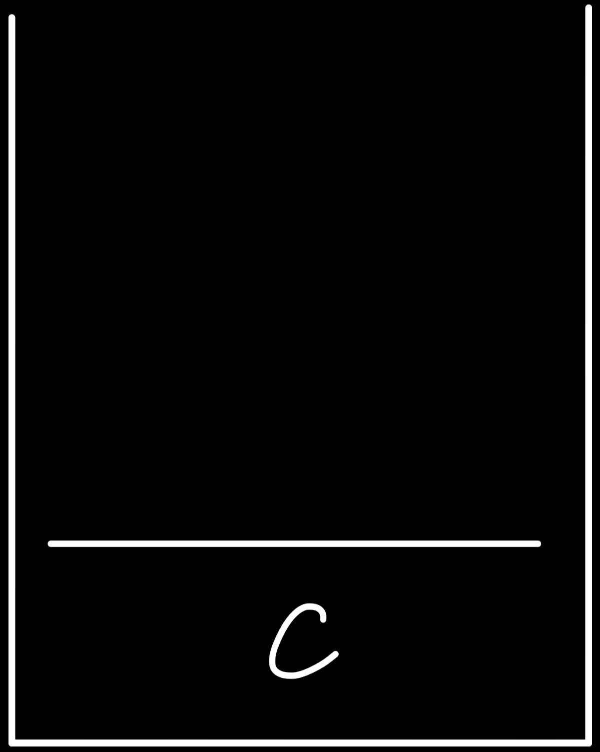
Push order: A B C

Pop order: A B C



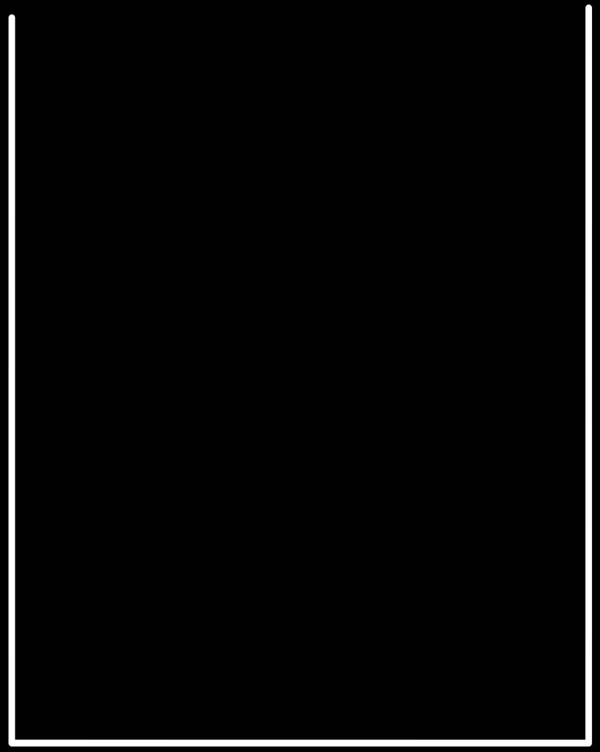
Push order: A B C

Pop order: A B C



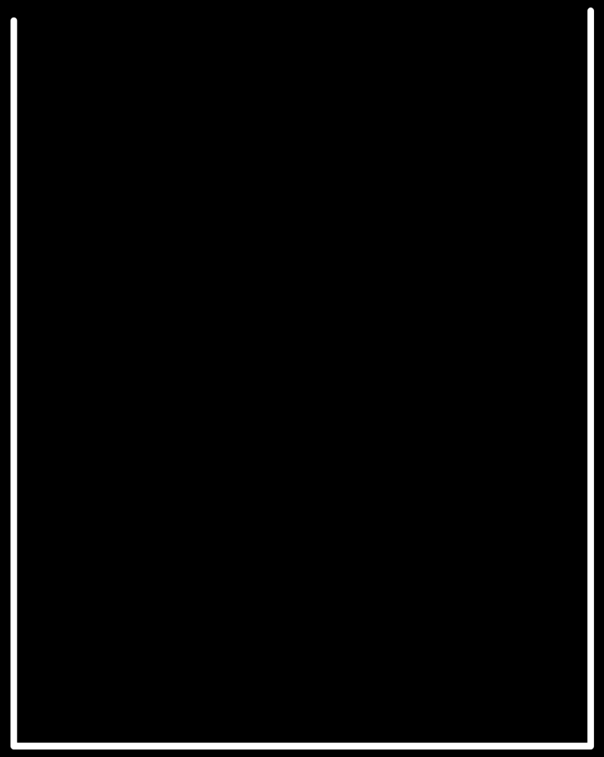
Push order: A B C

Pop order: A B C



Push order: A B C

Pop order: A B C



3 items — 5 possible pop order

- C B A
- A B C
- B A C
- B C A
- A C B

Push order: A B C

Pop order: A B C

3 items — 5 possible pop order

- C B A
- A B C
- B A C
- B C A
- A C B

4 items — 14 possible pop order

5 items — 42 possible pop order

Push order: A B C

Pop order: A B C

3 items — 5 possible pop order

- C B A
- A B C
- B A C
- B C A
- A C B

4 items — 14 possible pop order

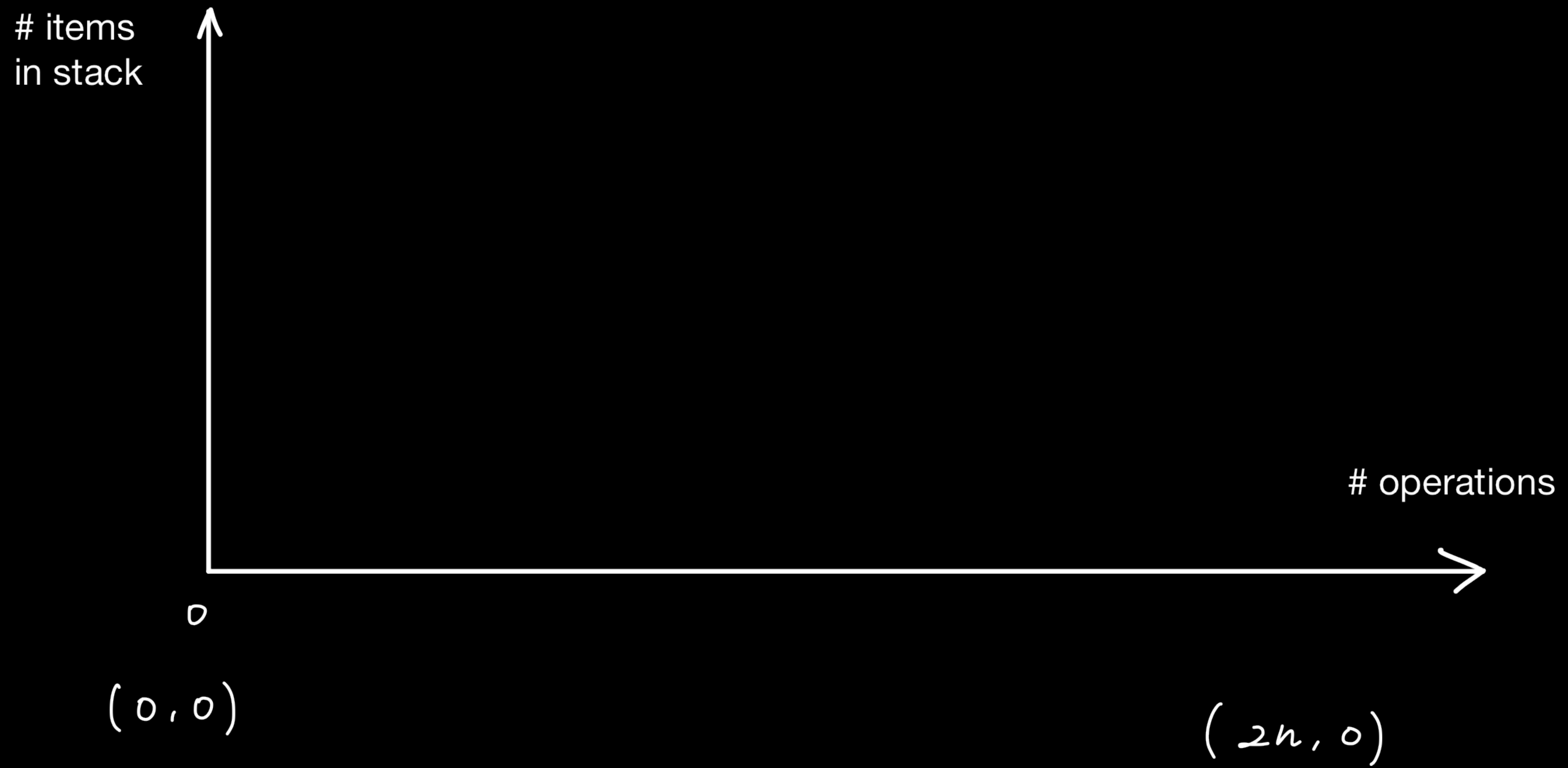
5 items — 42 possible pop order

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

n items, 2n operations

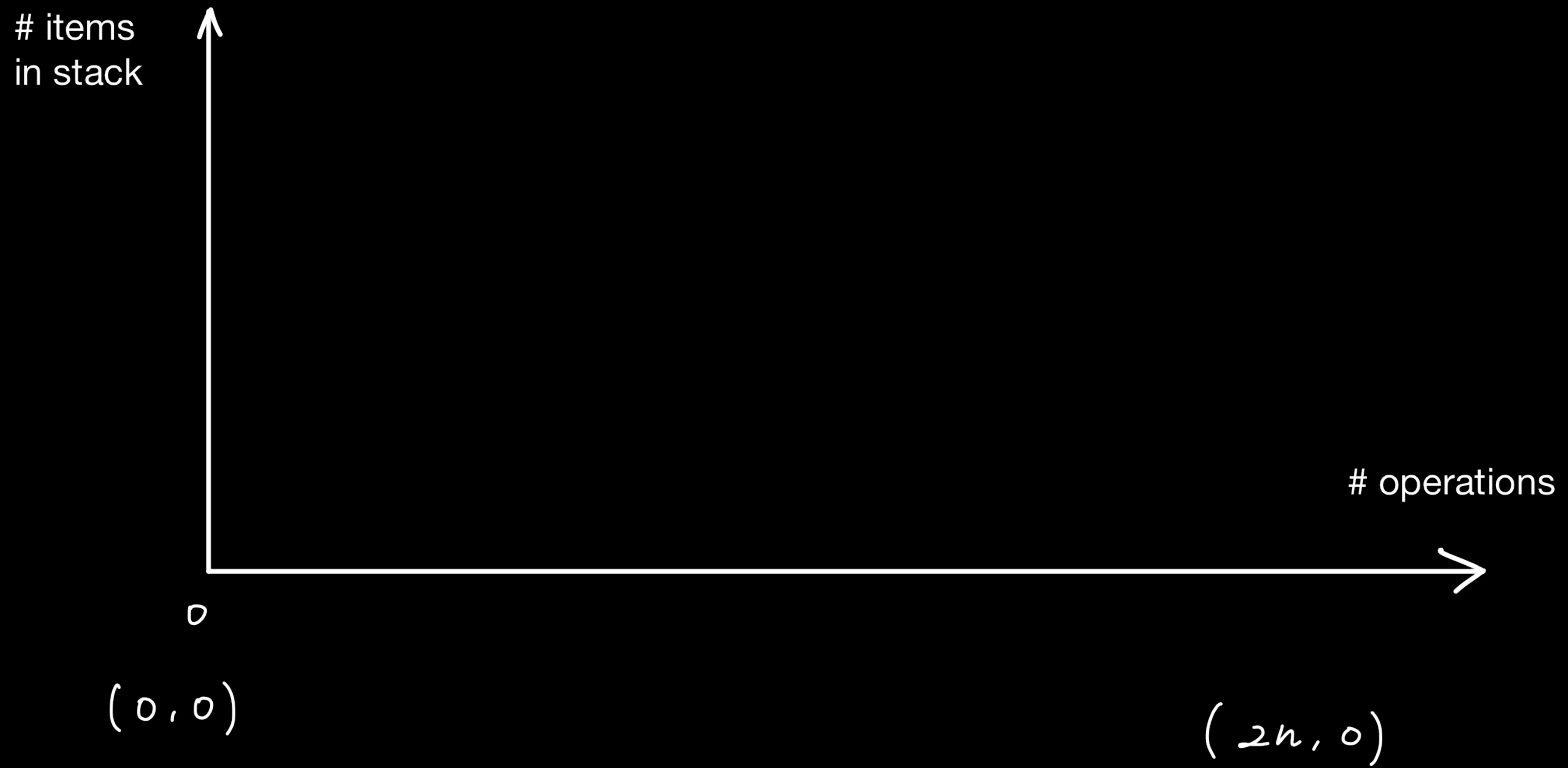
- n pushes
- n pops



$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

n items, 2n operations

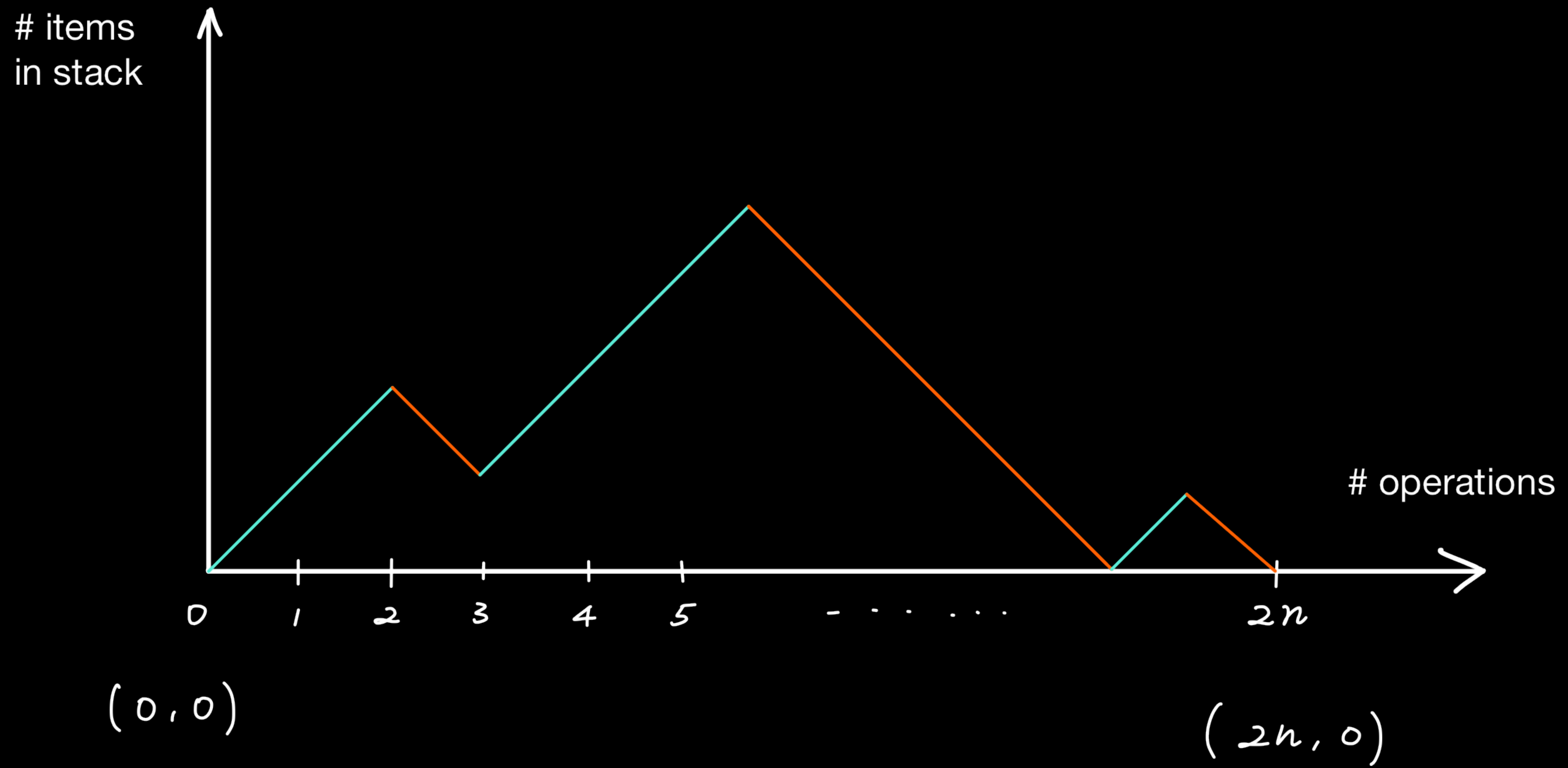
- n pushes
- n pops



$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

n items, 2n operations

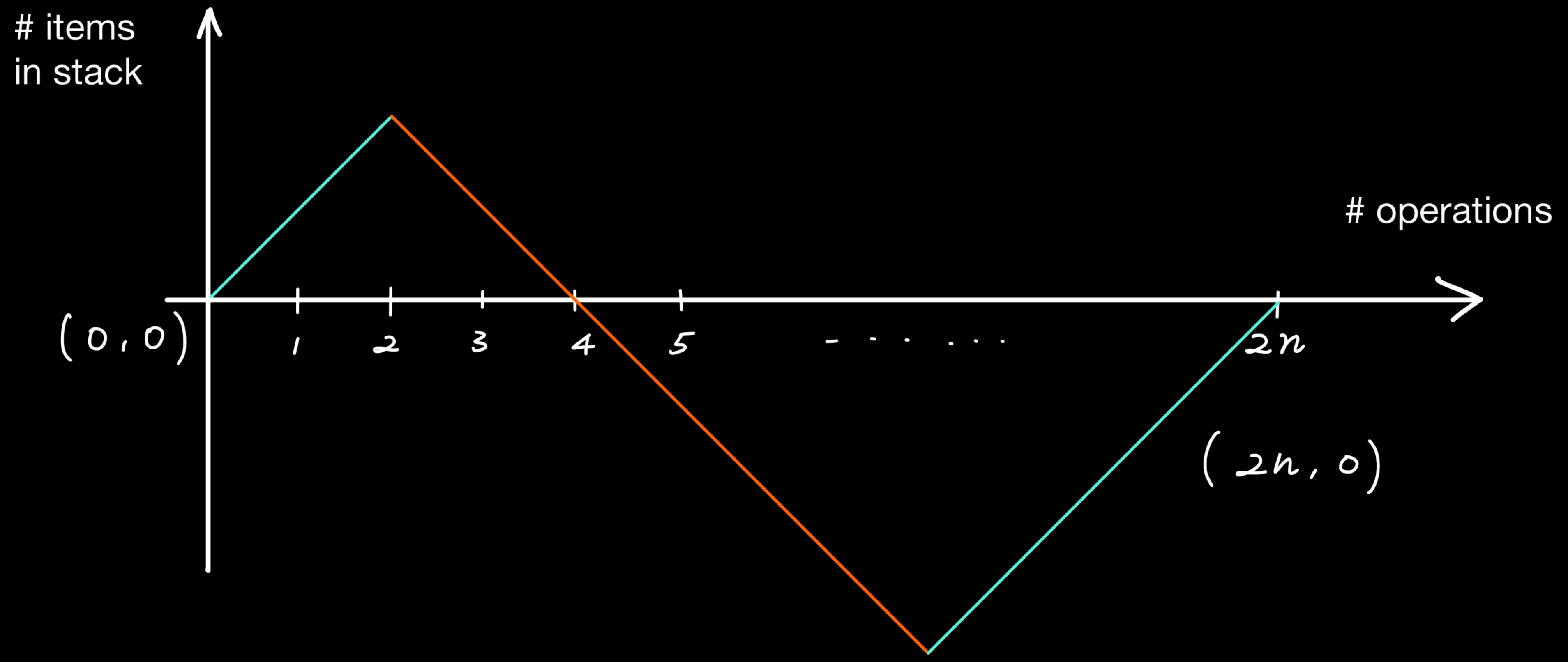
- n pushes
- n pops



$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

n items, 2n operations

- n pushes
- n pops



$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

n items, 2n operations

- n pushes
- n pops

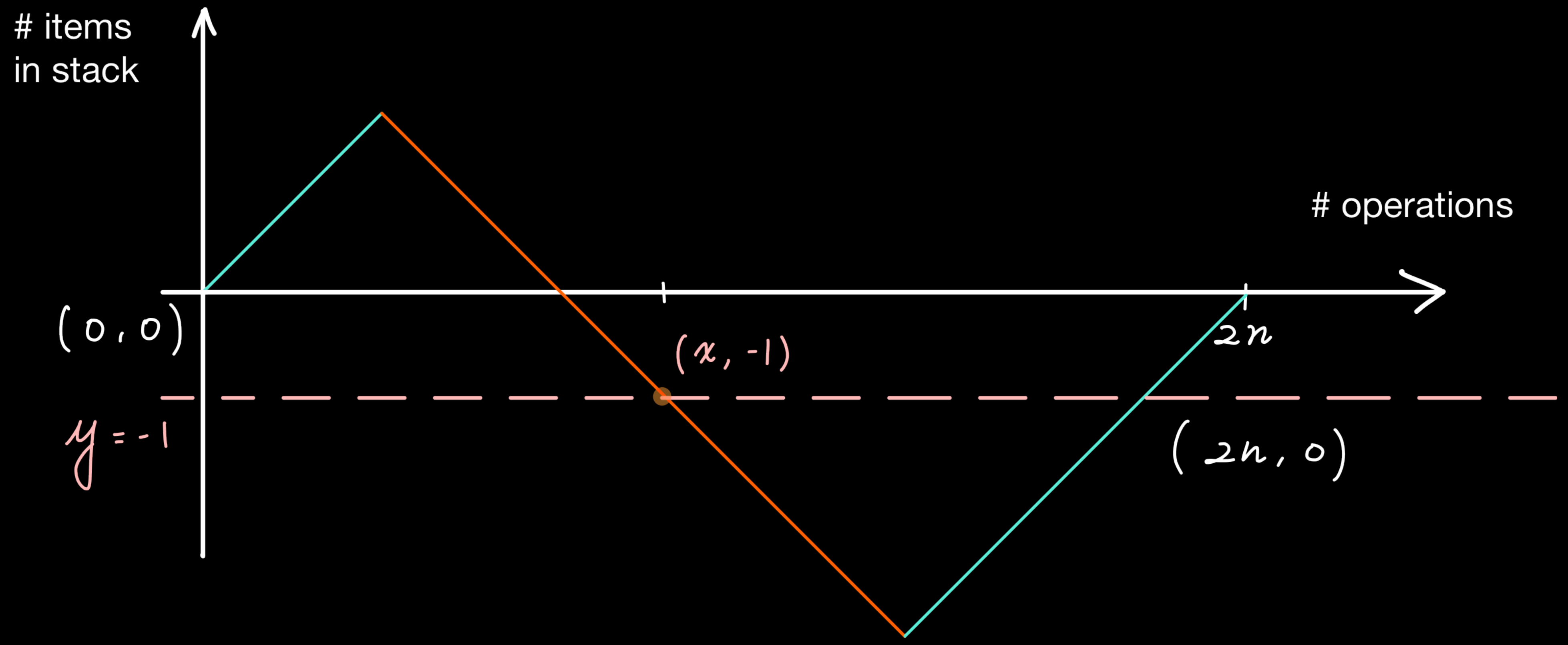
$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

n items, 2n operations

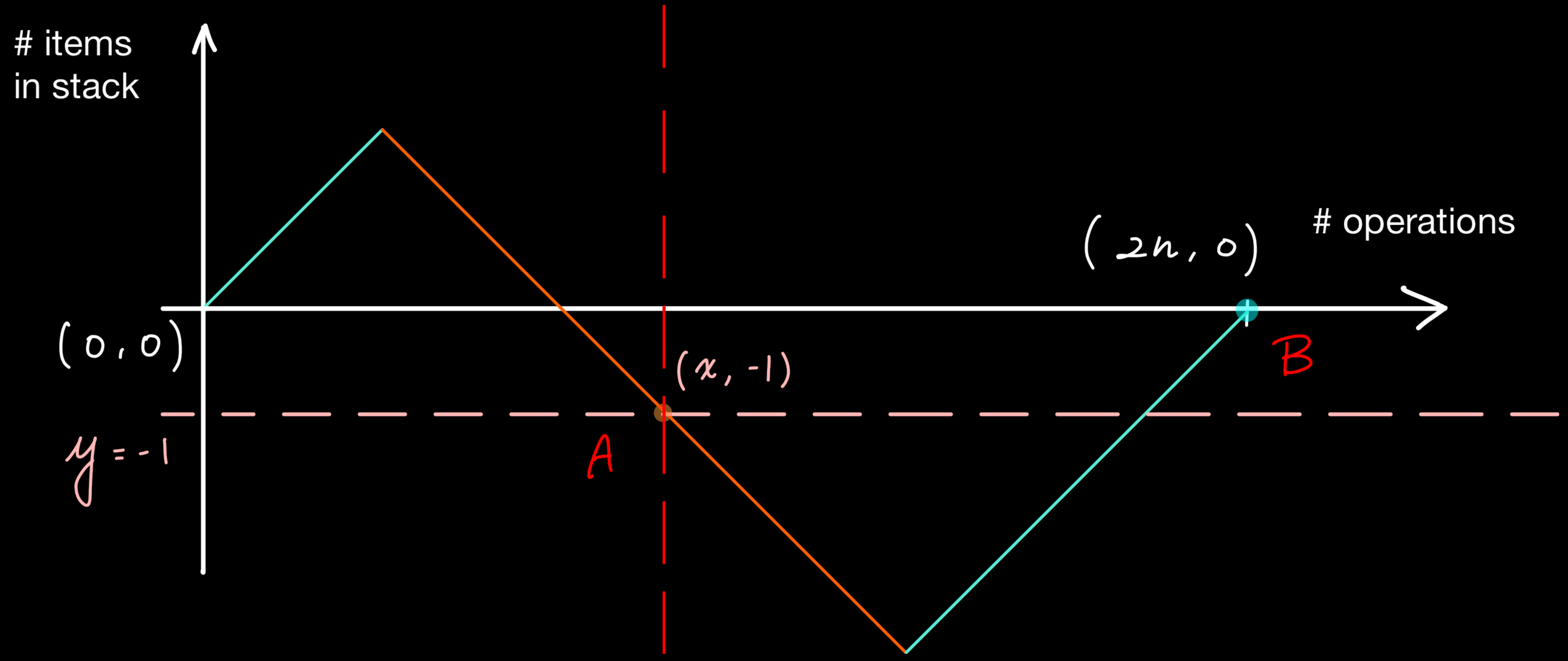
- n pushes
- n pops

$$C_n = \binom{2n}{n} - \#Illegal$$

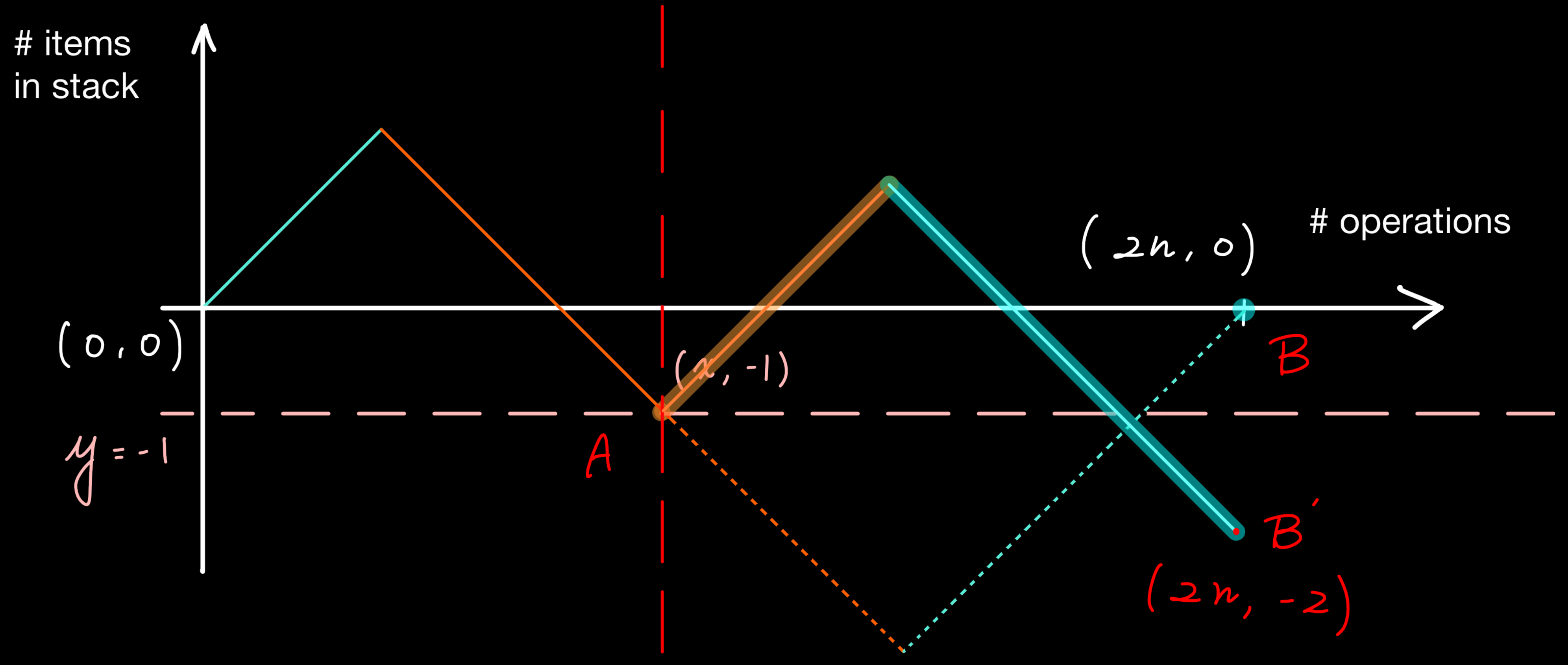
$$C_n = \binom{2n}{n} - \#Illegal$$



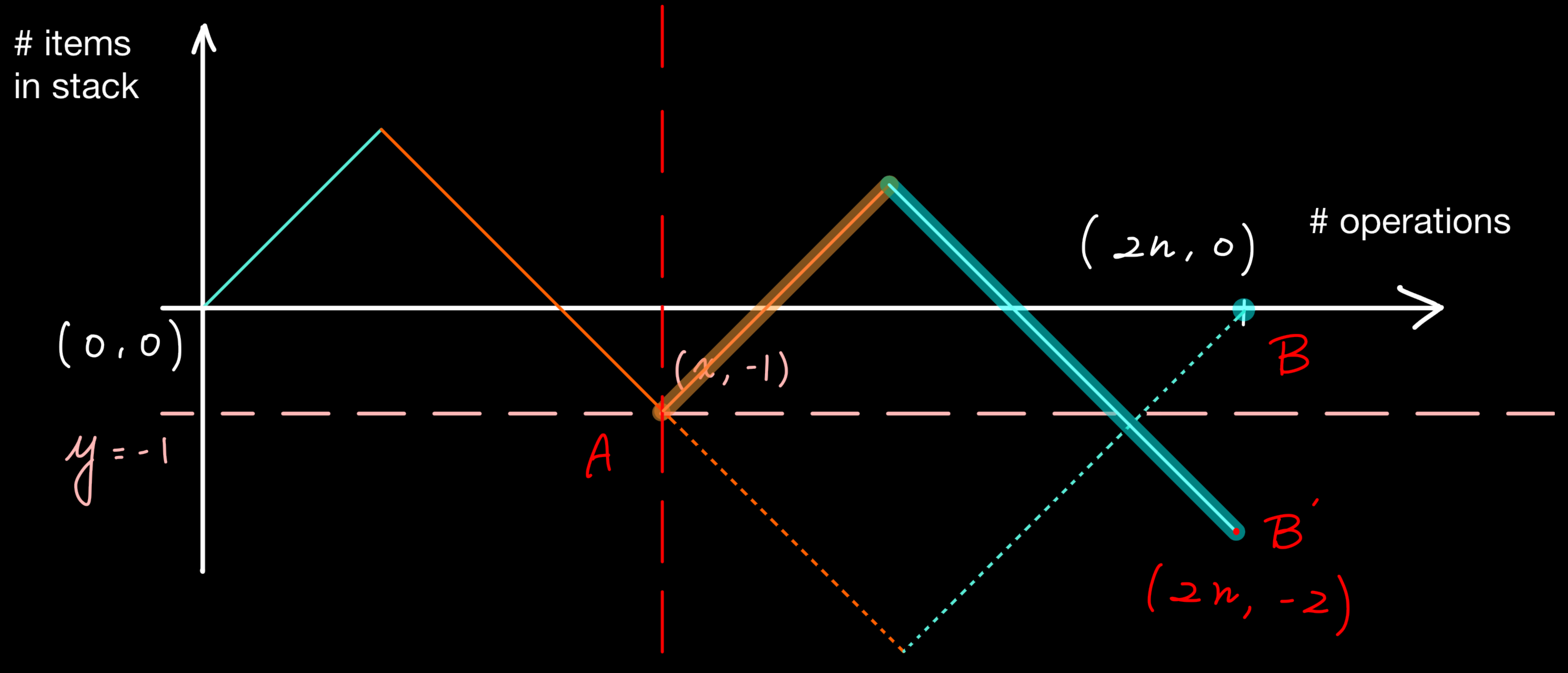
$$C_n = \binom{2n}{n} - \#Illegal$$



$$C_n = \binom{2n}{n} - \#Illegal$$



$$C_n = \binom{2n}{n} - \#Illegal$$



$$\#Illegal = \#path(0,0) \longrightarrow (2n,-2) = \binom{2n}{n+1}$$

$$C_n = \binom{2n}{n} - \#Illegal$$

$$= \binom{2n}{n} - \binom{2n}{n+1}$$

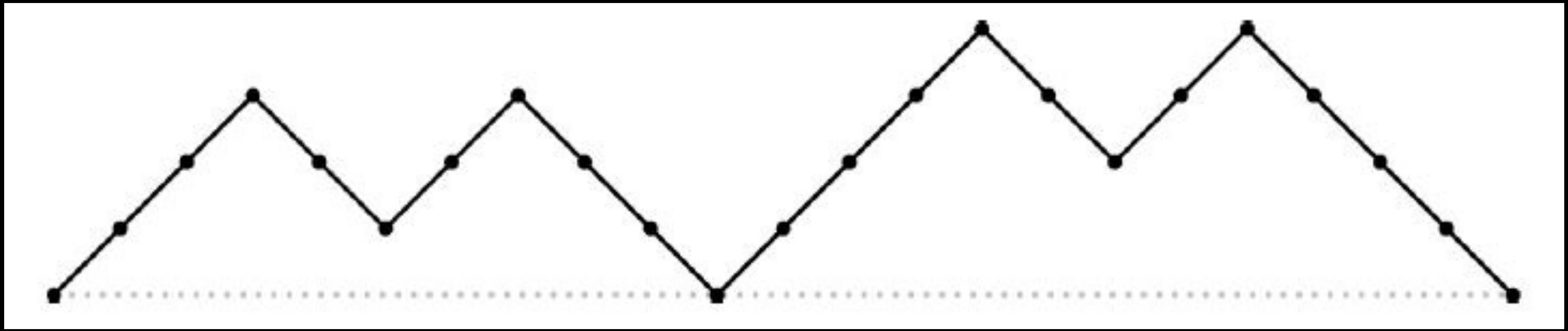
$$= \frac{2n(2n-1)...(n+1)}{n!} - \frac{2n(2n-1)...(n+1)n}{(n+1)!}$$

$$= \frac{2n(2n-1)...(n+1)(n+1) - 2n(2n-1)...(n+1)n}{(n+1)!}$$

$$= \frac{(n+1-n)2n(2n-1)...(n+1)}{(n+1)(n!)}$$

$$= \frac{1}{n+1}\binom{2n}{n}$$

Dyck Path



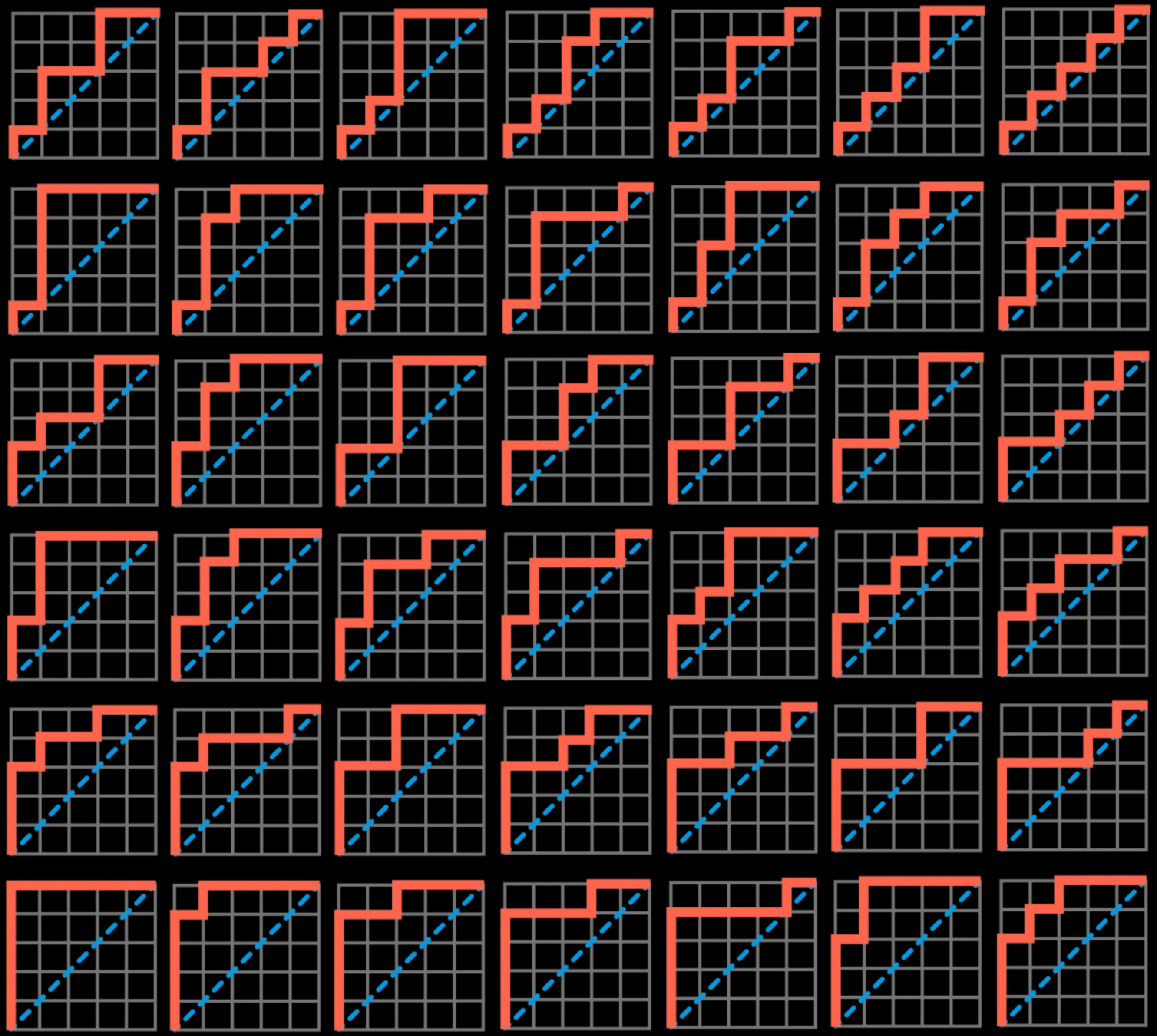
$$step \in \{(1,1),(1,-1)\}$$

$$path : (0,0) \longrightarrow (2n,0)$$

Dyck Path

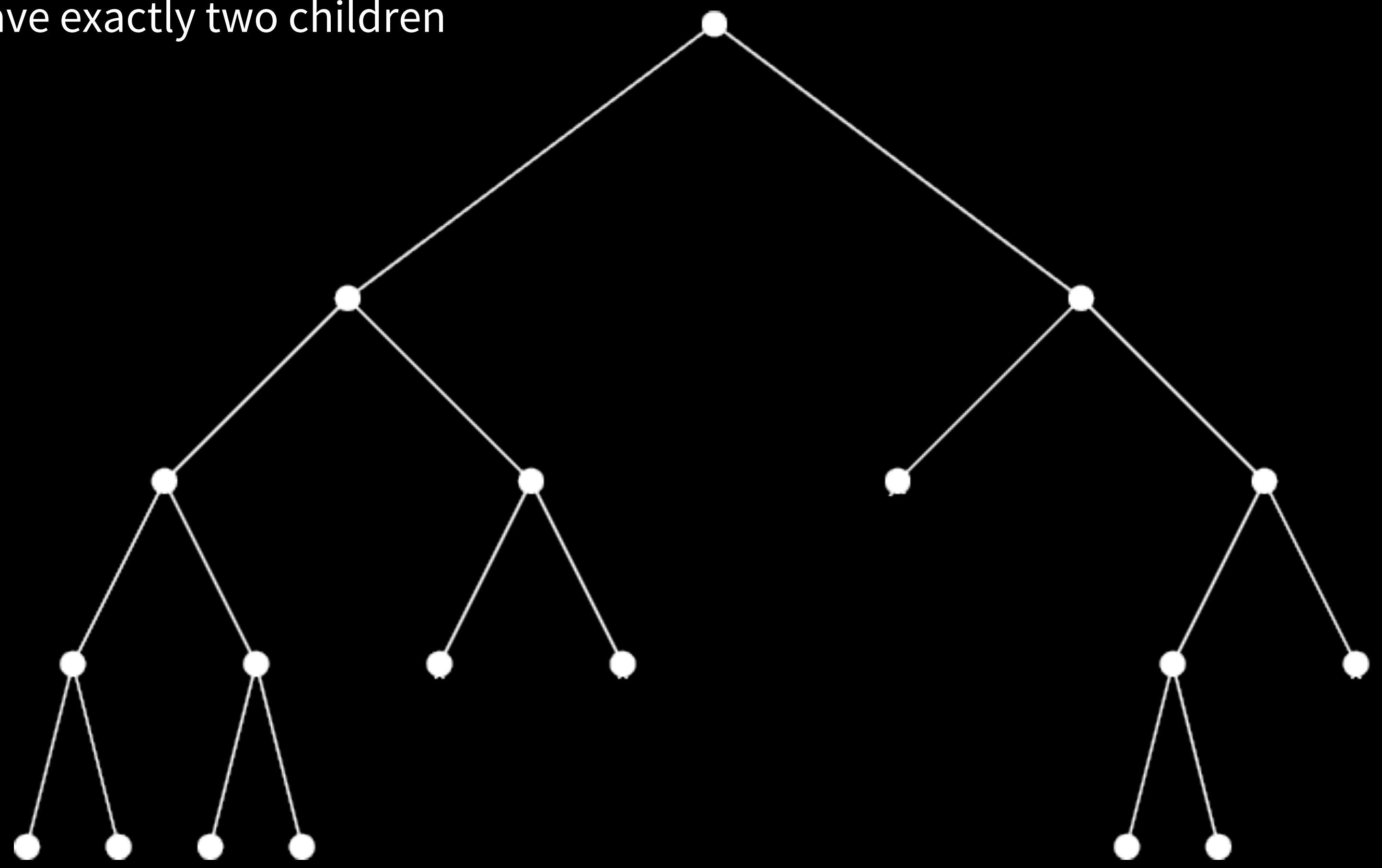
$$path : (0,0) \longrightarrow (n,n)$$

$$step \in \{(1,0),(0,1)\}$$



Proper Binary Tree

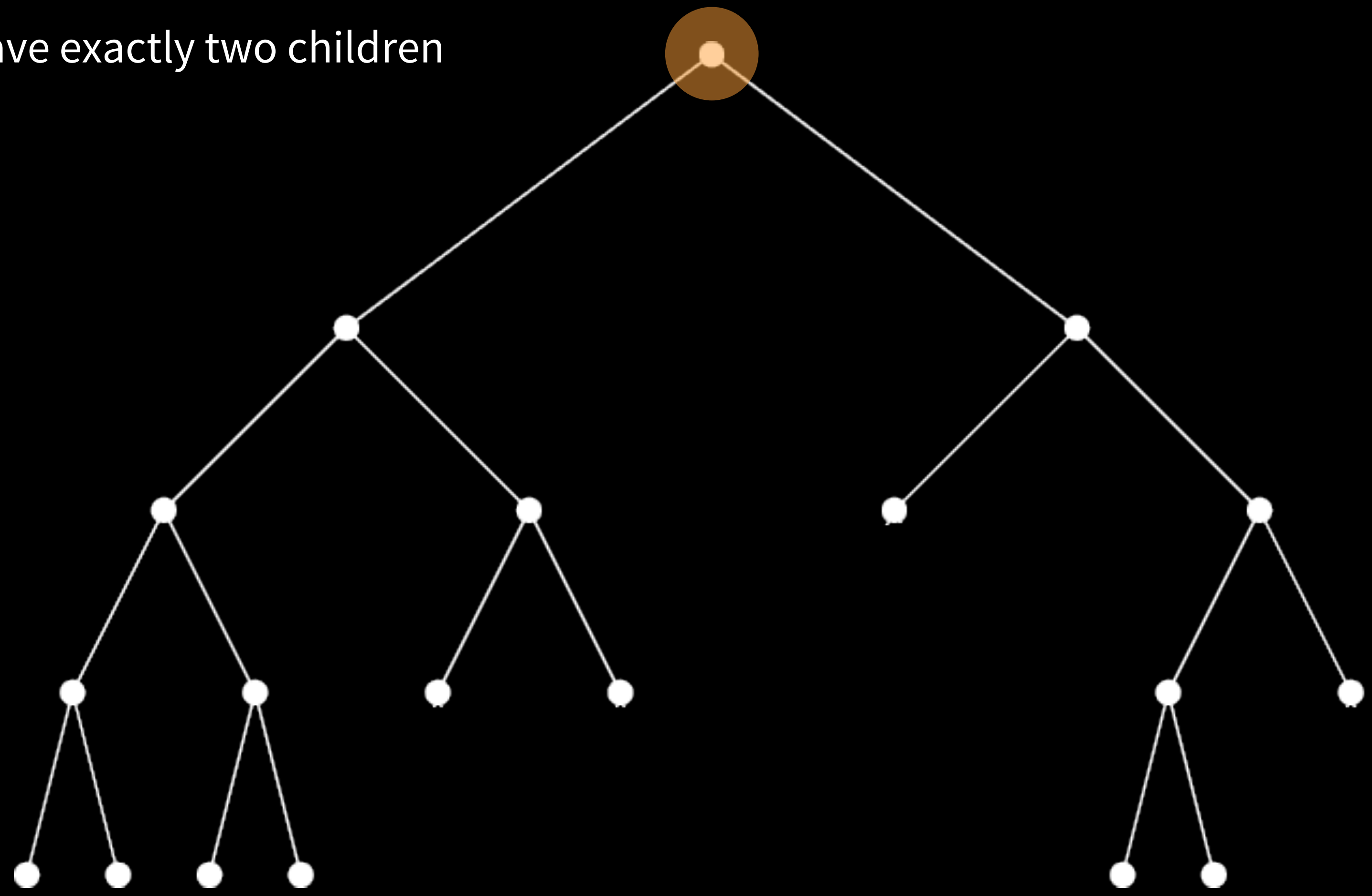
All Internal Nodes (not leaf) have exactly two children



Proper Binary Tree

All Internal Nodes (not leaf) have exactly two children

Has a root node (deg = 2)

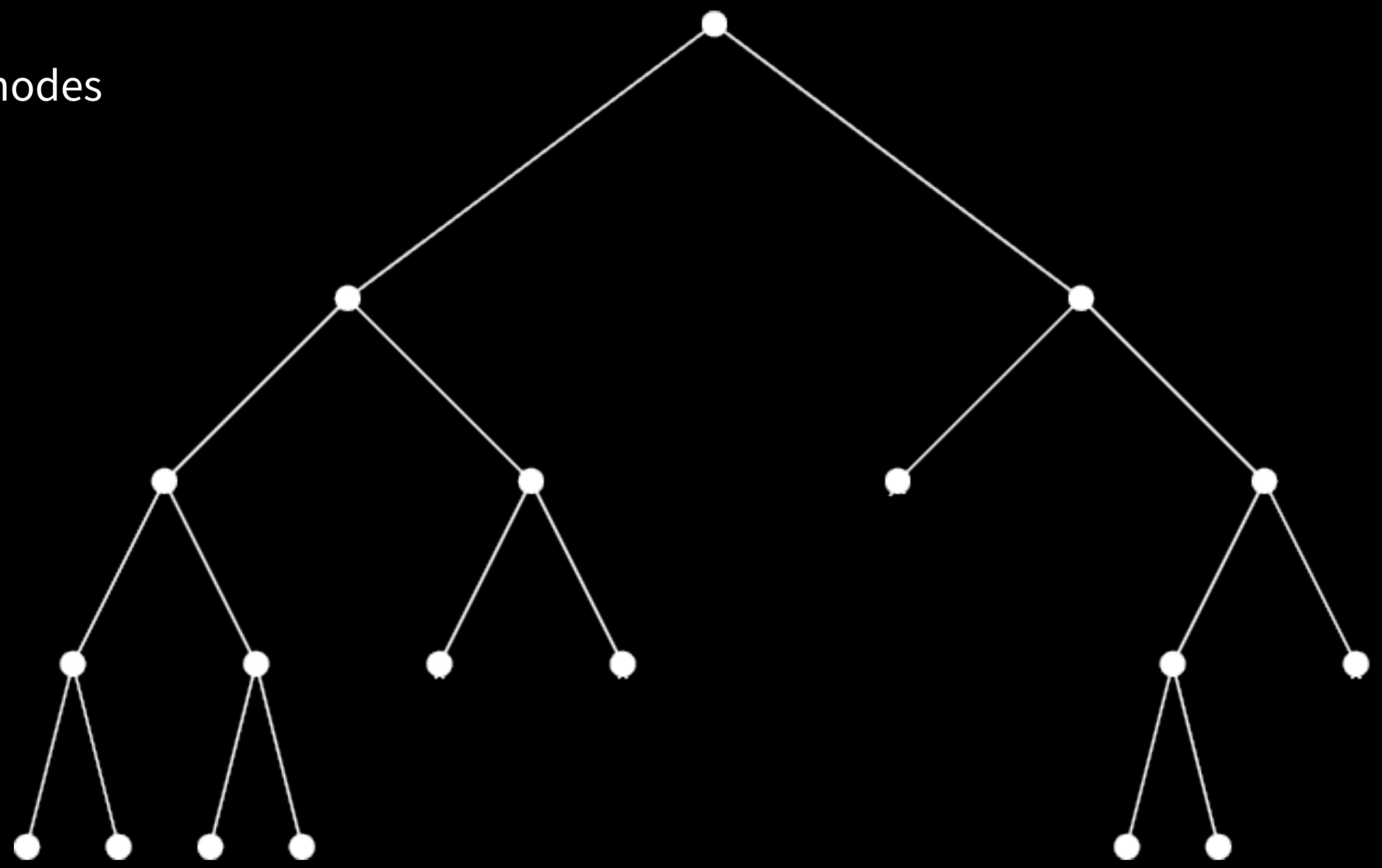


Bijection

Proper binary tree with n internal nodes

$\longleftrightarrow$

Dyck path length 2n

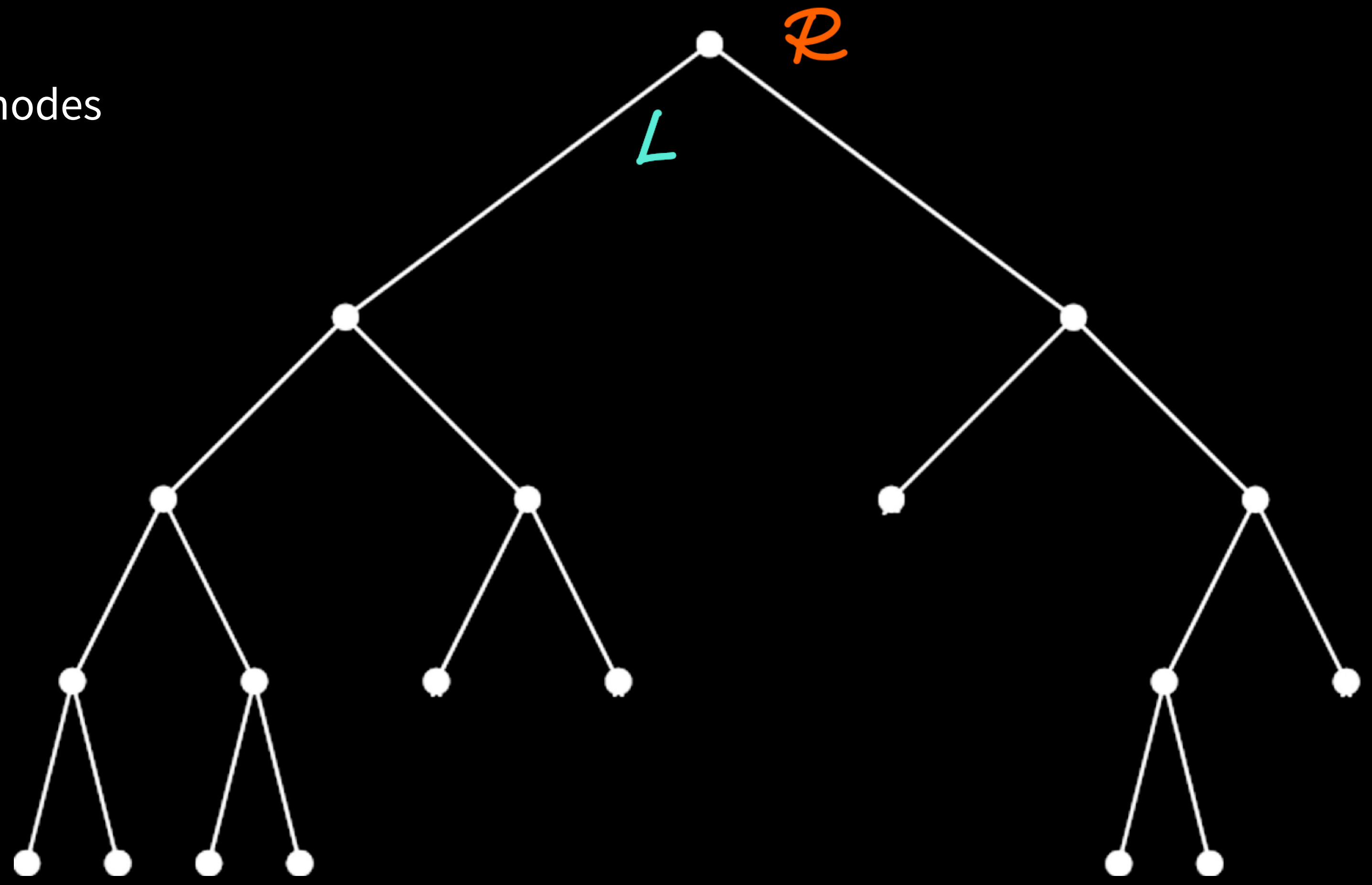


Bijection

Proper binary tree with n internal nodes

$\longleftrightarrow$

Dyck path length 2n

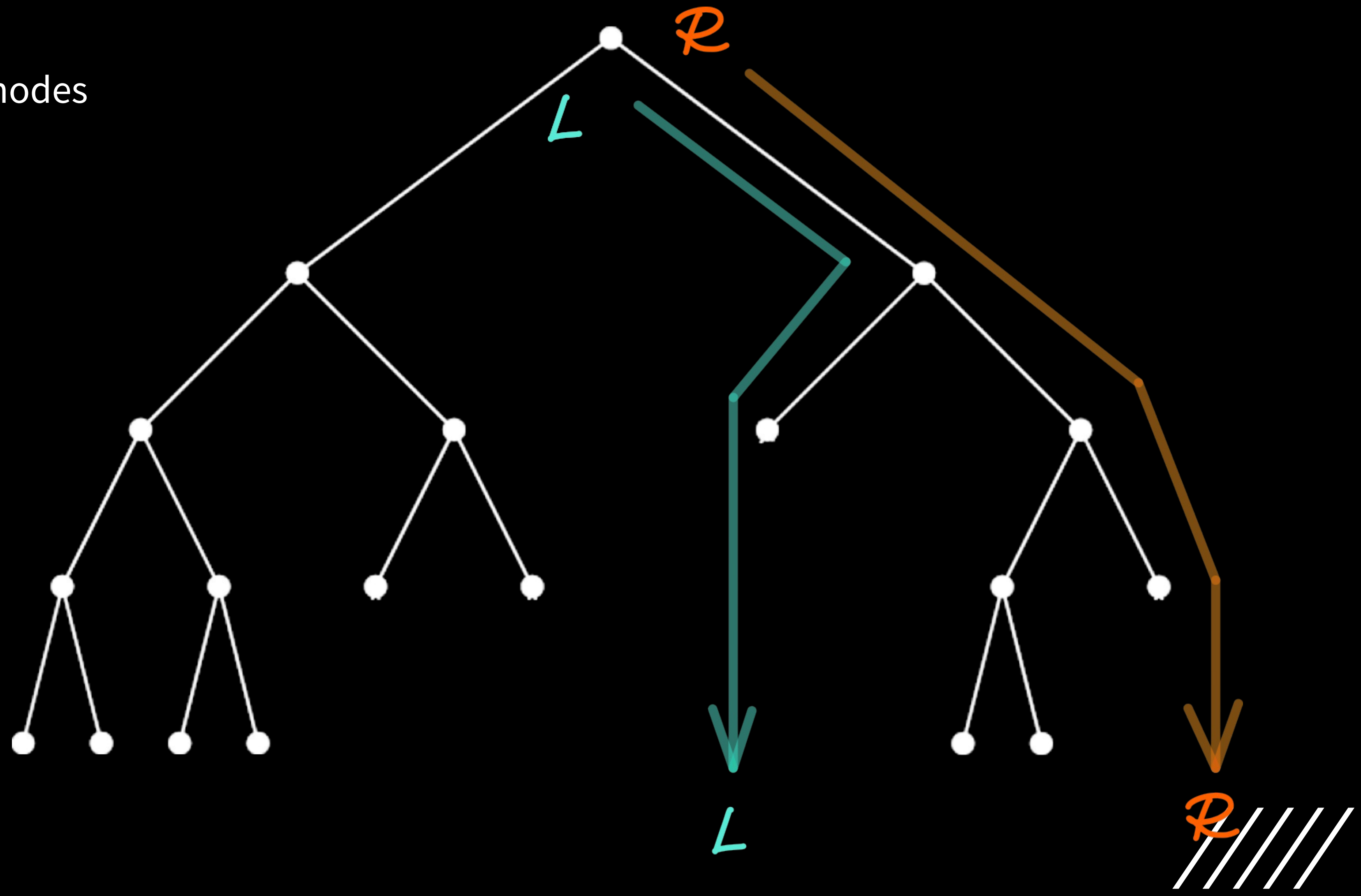


Bijection

Proper binary tree with n internal nodes

$\longleftrightarrow$

Dyck path length 2n

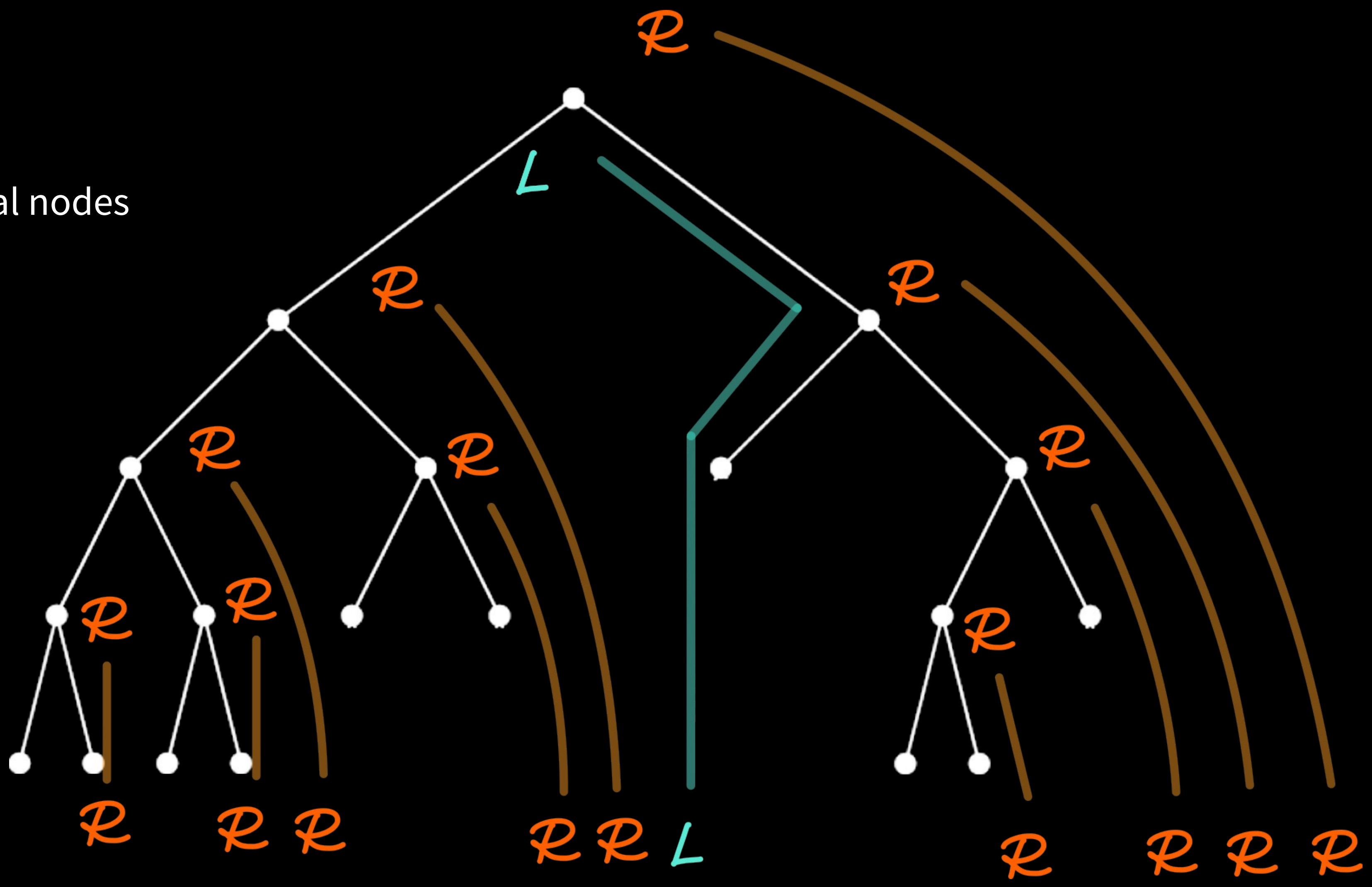


Bijection

Proper binary tree with n internal nodes

$\longleftrightarrow$

Dyck path length 2n

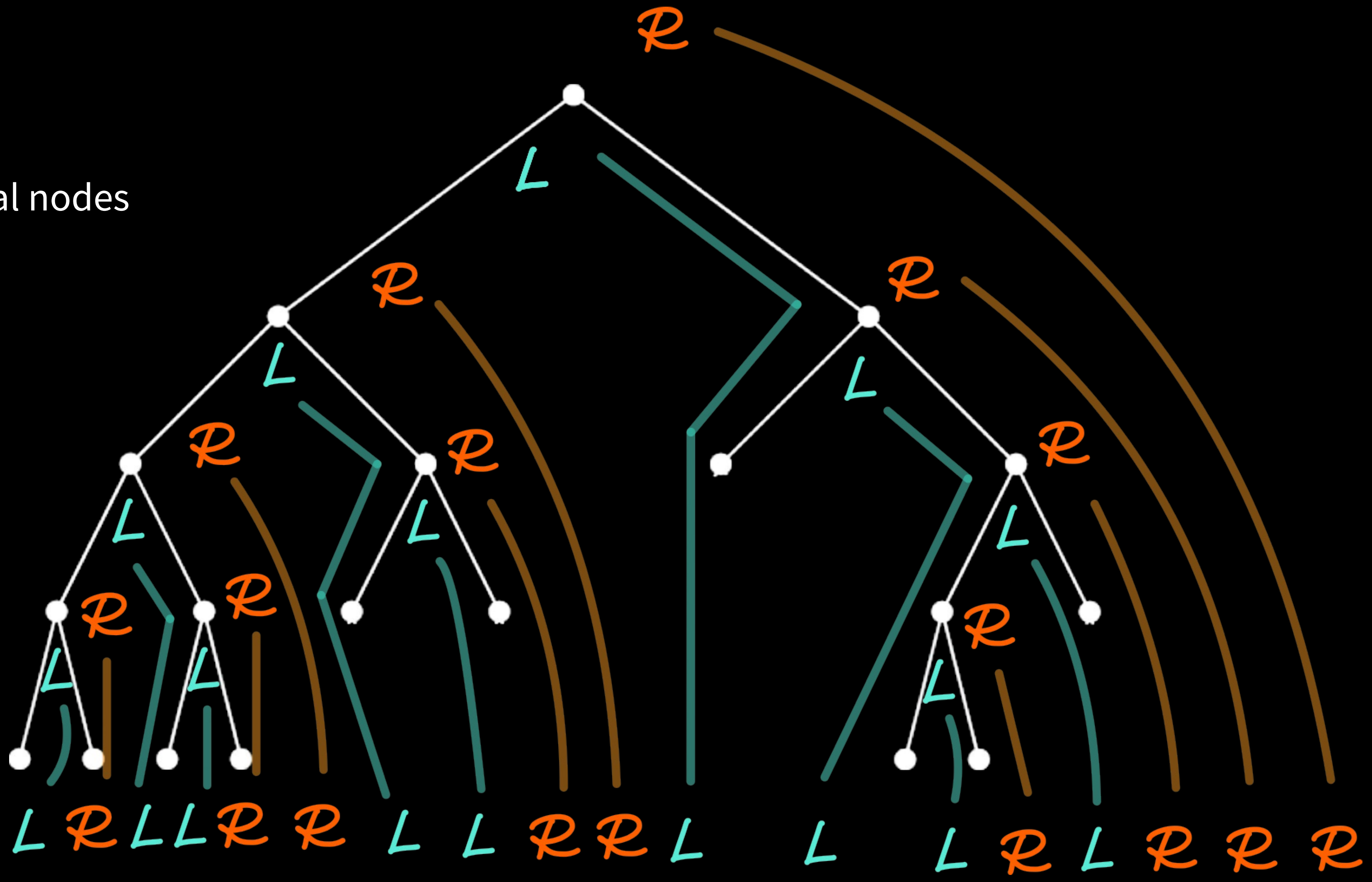


Bijection

Proper binary tree with n internal nodes

$\longleftrightarrow$

Dyck path length 2n

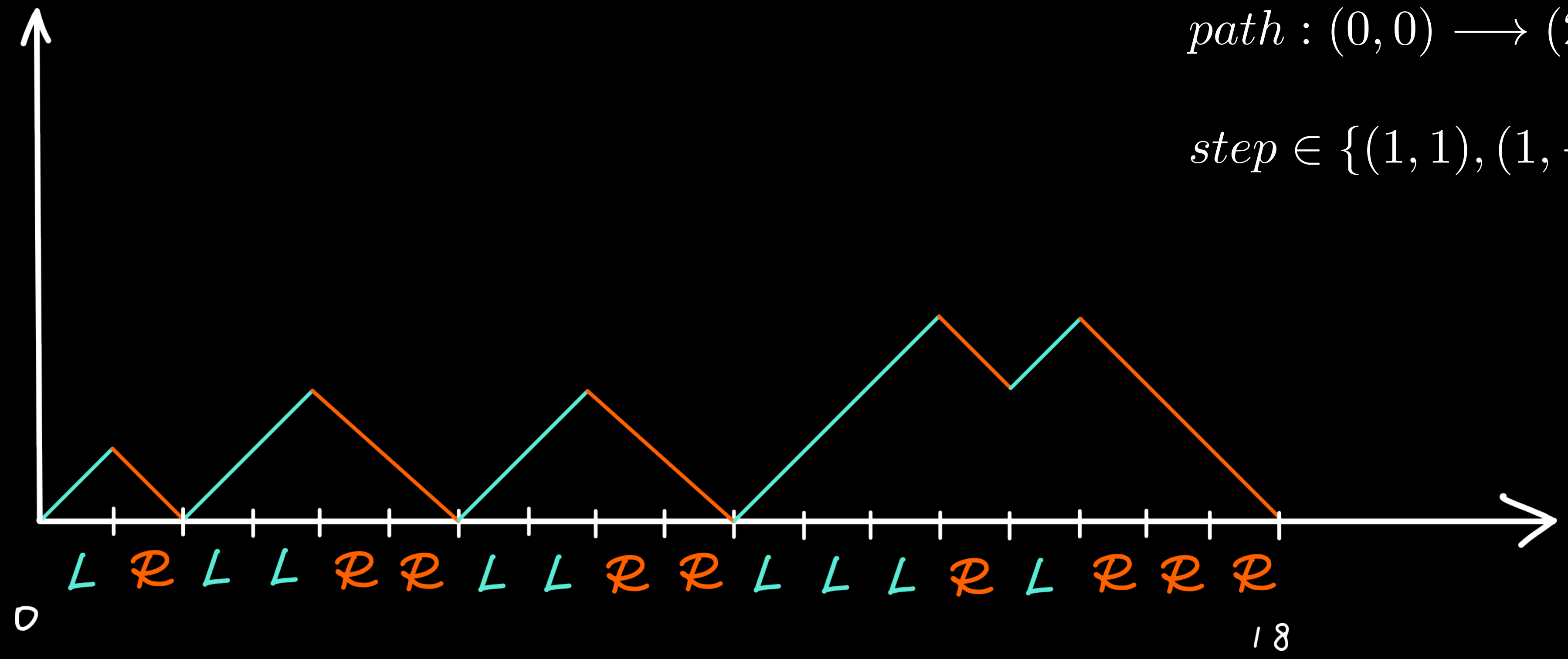


Bijection

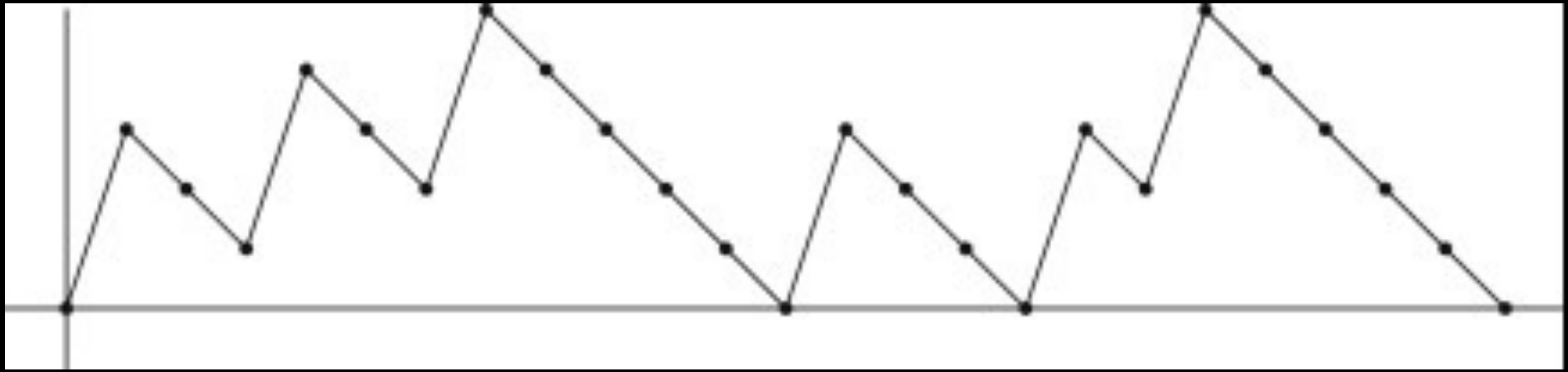
Proper binary tree with n internal nodes $\longleftrightarrow$ Dyck path length 2n

$$path : (0,0) \longrightarrow (2n, 0)$$

$$step \in \{(1,1), (1,-1)\}$$



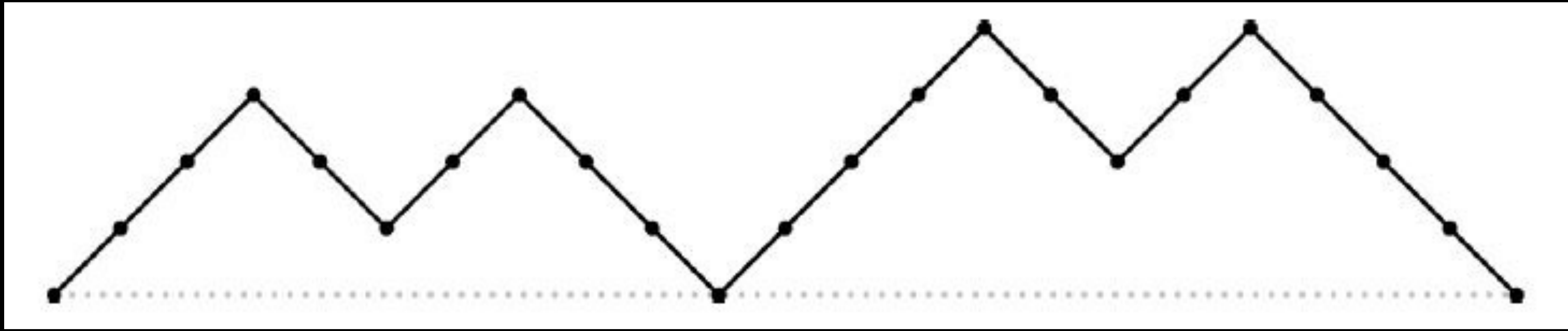
Generalised Dyck Path



$$step \in \{(1,k),(1,-1)\}$$

$$path : (0,0) \longrightarrow (?,0)$$

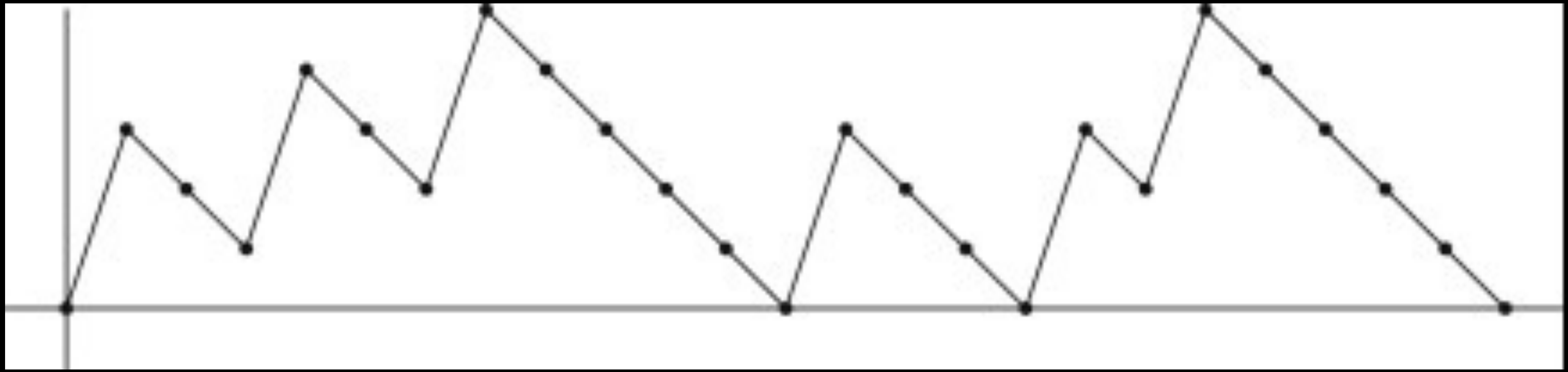
Dyck Path



$$step \in \{(1,1),(1,-1)\}$$

$$path : (0,0) \longrightarrow (2n,0) = ((1+1)n,0)$$

Generalised Dyck Path

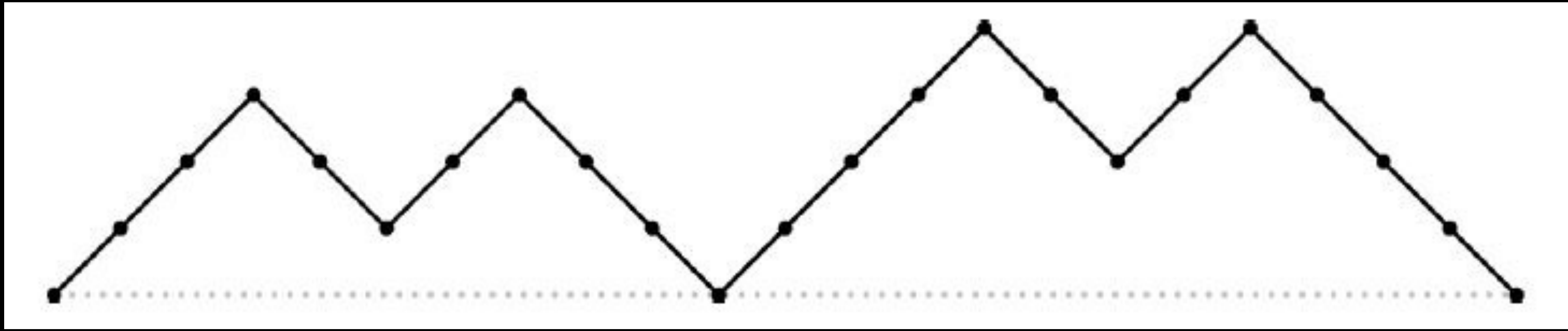


$$step \in \{(1,k),(1,-1)\}$$

$$path : (0,0) \longrightarrow ((k+1)n, 0)$$

Dyck Path

$$C_n = \frac{1}{n+1}\binom{(1+1)n}{n}$$

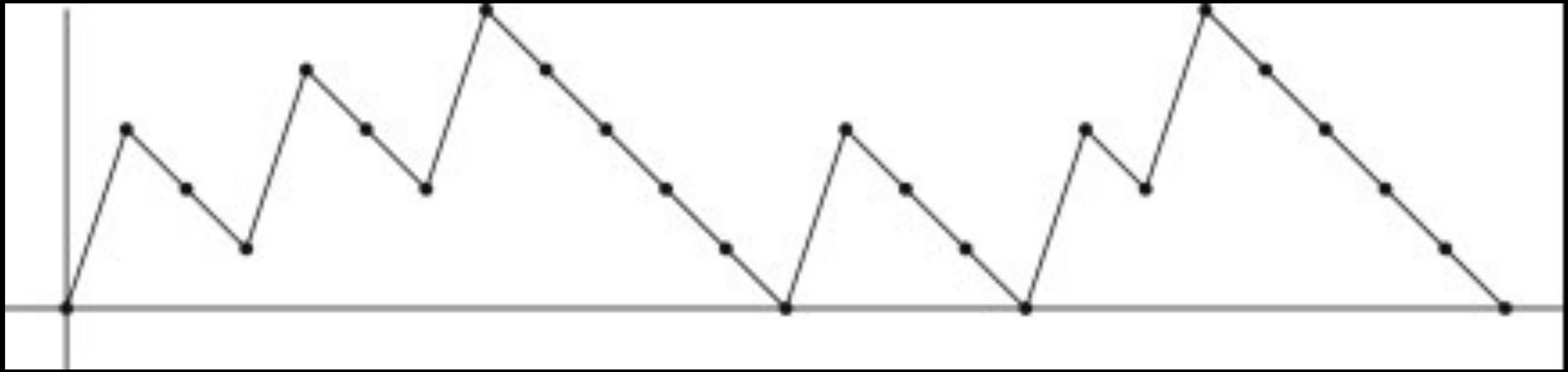


$$step \in \{(1,1),(1,-1)\}$$

$$path : (0,0) \longrightarrow (2n,0) = ((1+1)n, 0)$$

Generalised Dyck Path

$$C_n = \frac{1}{kn+1}\binom{(k+1)n}{n}$$

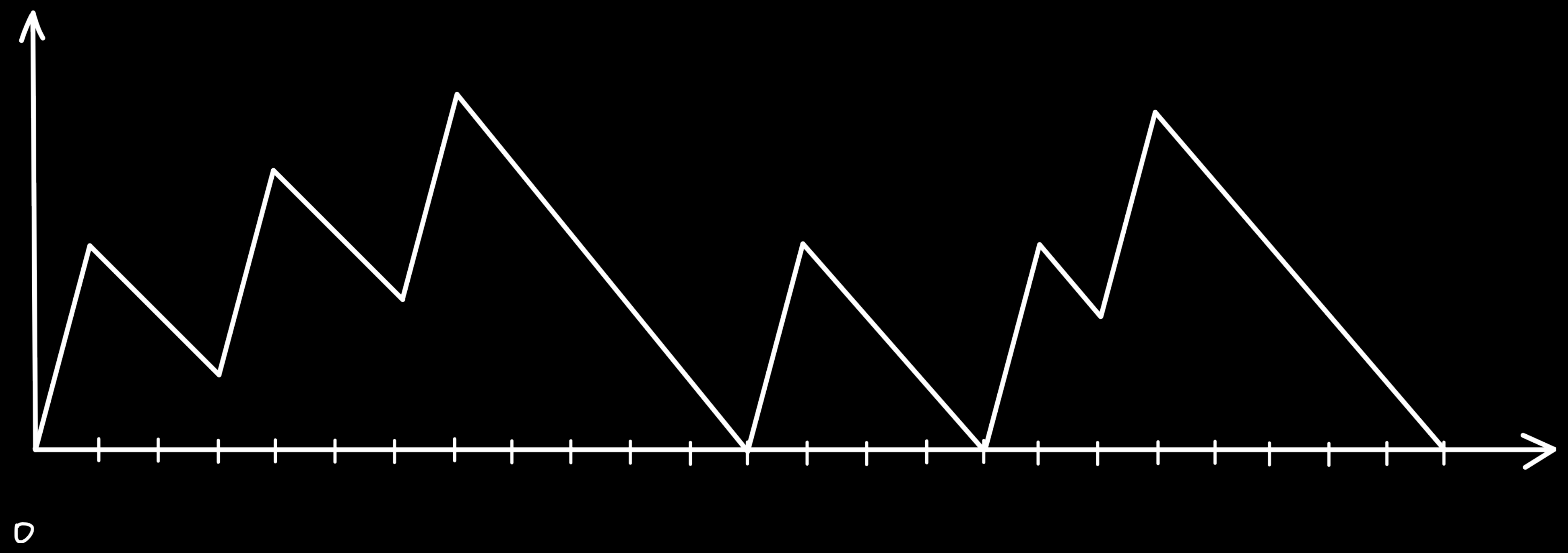


$$step \in \{(1,k),(1,-1)\}$$

$$path : (0,0) \longrightarrow ((k+1)n, 0)$$

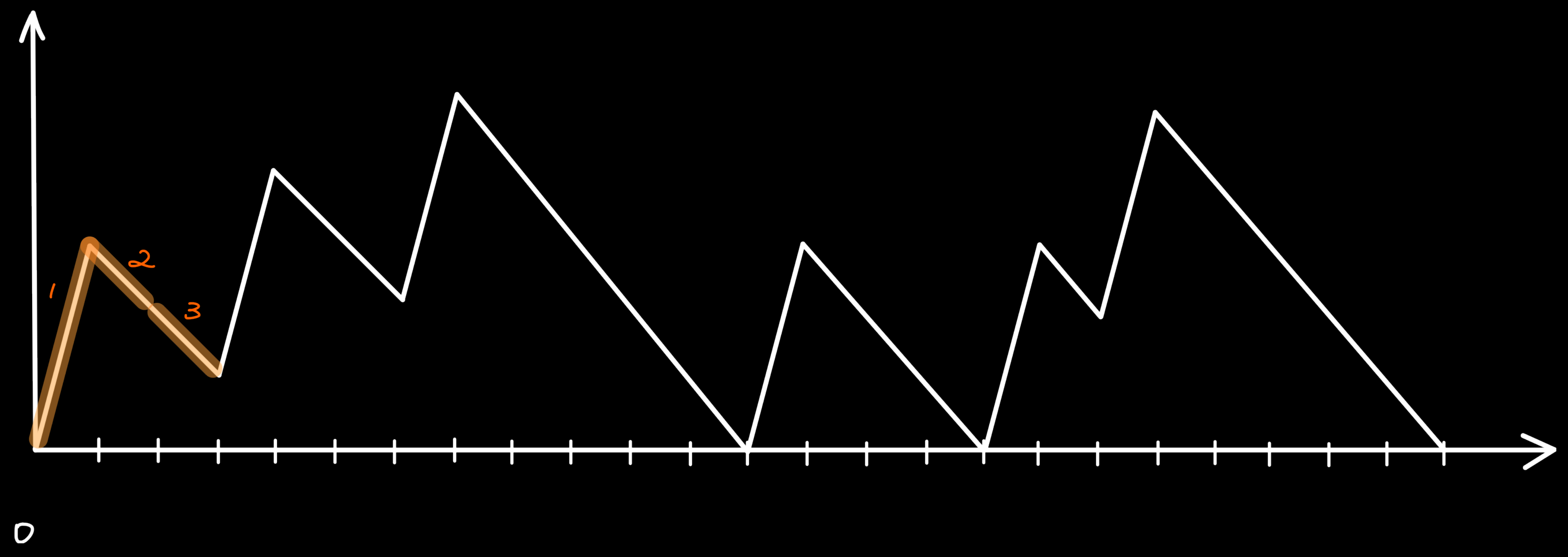
Bijection 2.0

Proper (k+1)-nary tree $\longleftrightarrow$ Generalised (kth) Dyck Path



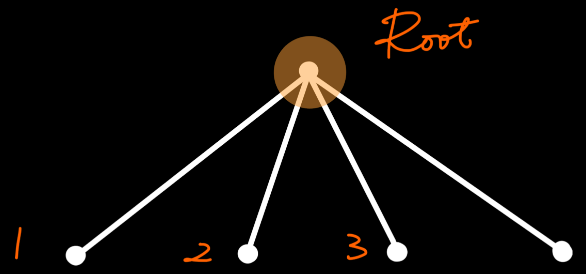
Bijection 2.0

Proper (k+1)-nary tree ⟷ Generalised (kth) Dyck Path



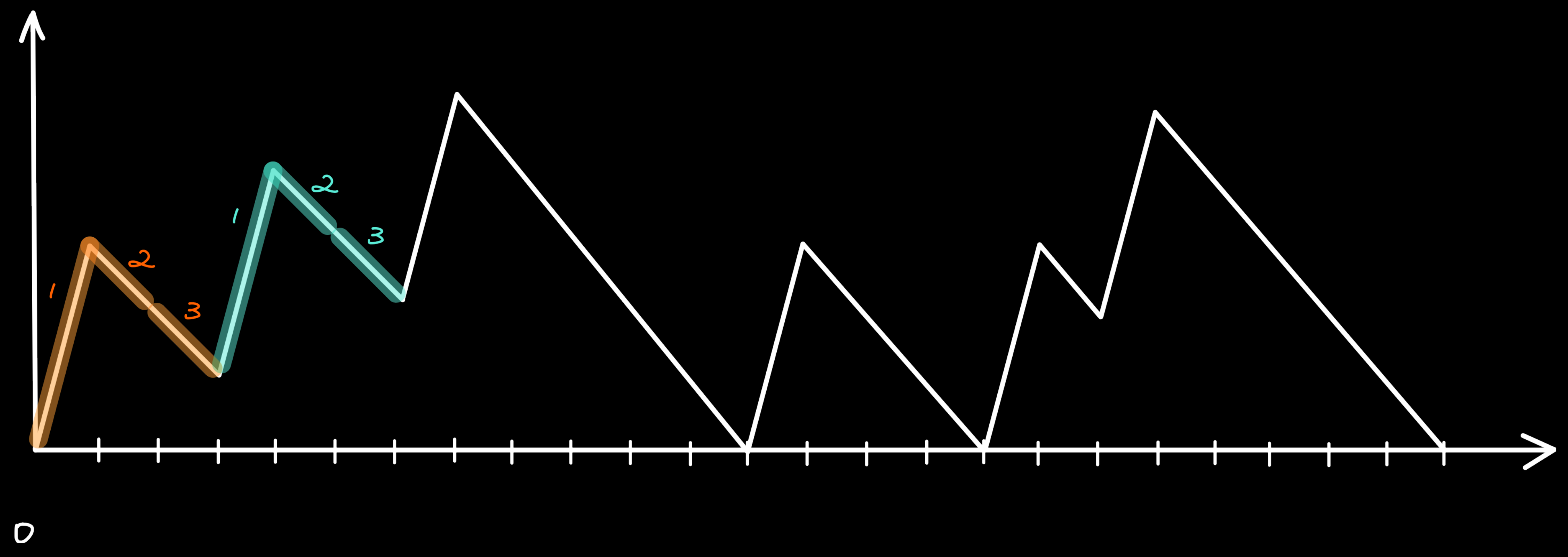
Bijection 2.0

Proper (k+1)-nary tree ⟷ Generalised (kth) Dyck Path



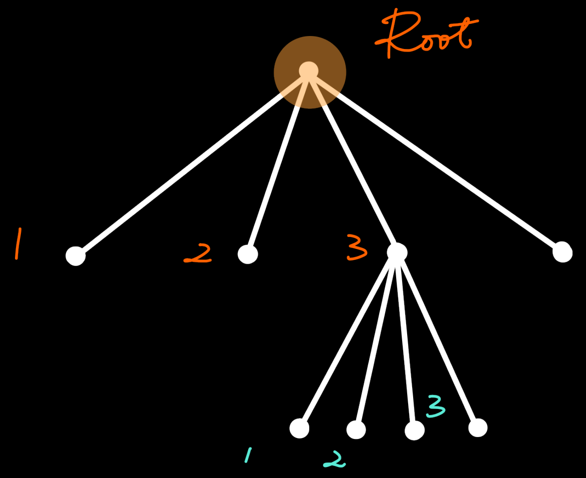
Bijection 2.0

Proper (k+1)-nary tree ⟷ Generalised (kth) Dyck Path



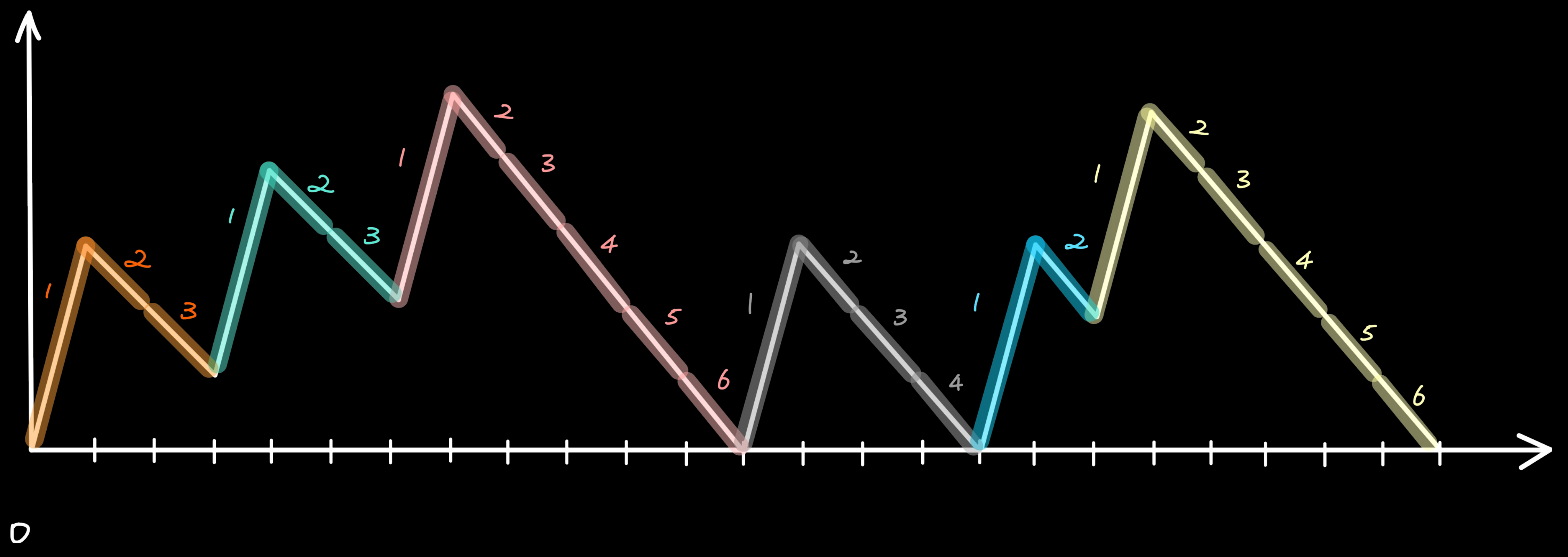
Bijection 2.0

Proper (k+1)-nary tree ⟷ Generalised (kth) Dyck Path



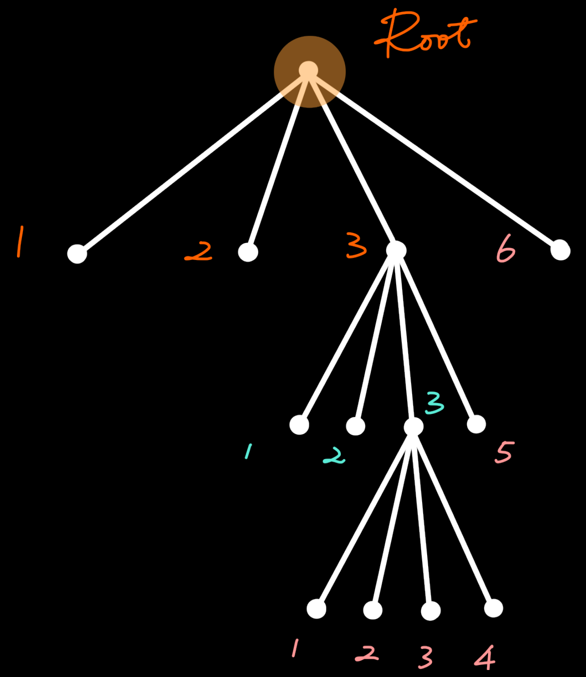
Bijection 2.0

Proper (k+1)-nary tree ⟷ Generalised (kth) Dyck Path



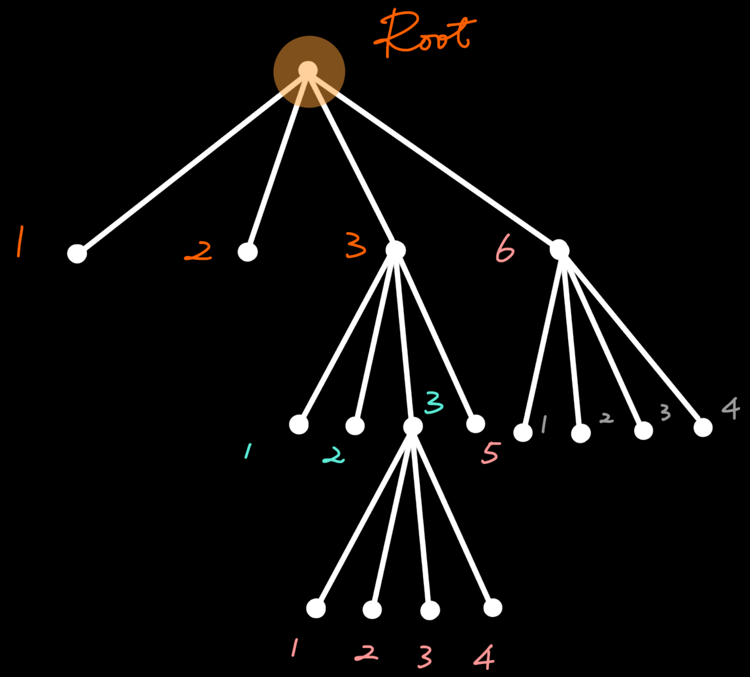
Bijection 2.0

Proper (k+1)-nary tree ⟷ Generalised (kth) Dyck Path



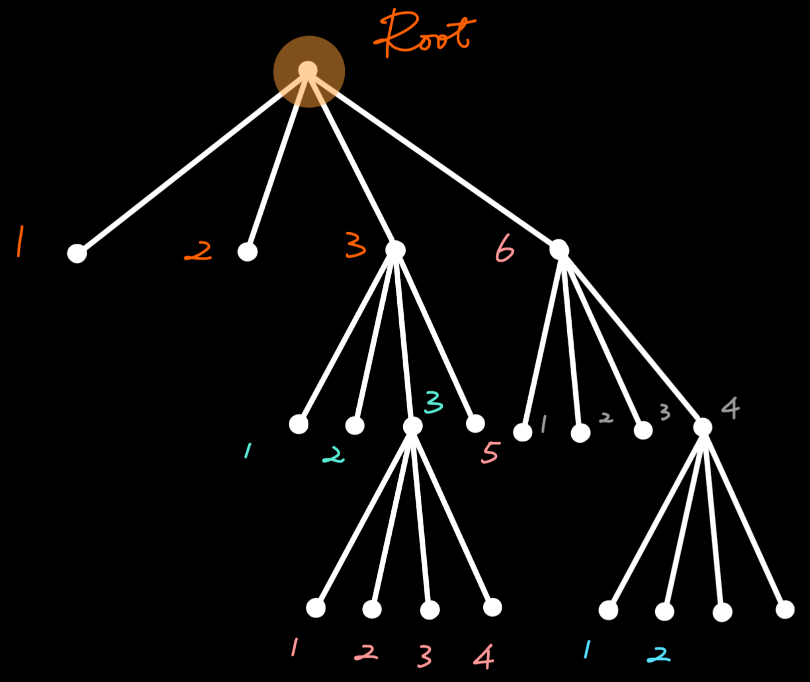
Bijection 2.0

Proper (k+1)-nary tree ⟷ Generalised (kth) Dyck Path



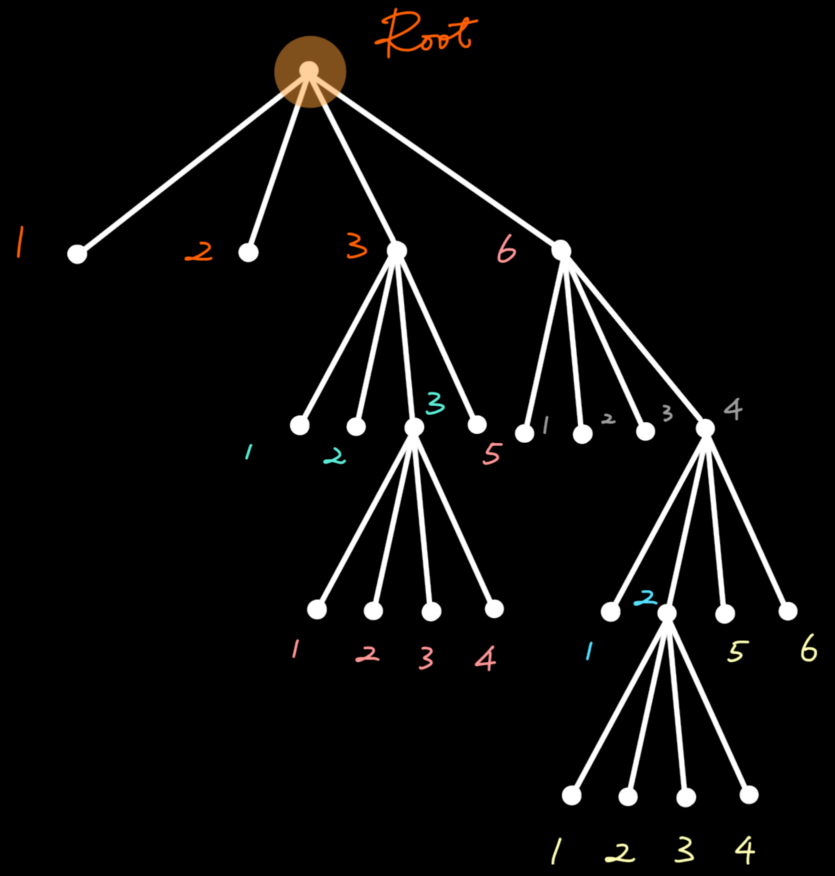
Bijection 2.0

Proper (k+1)-nary tree ⟷ Generalised (kth) Dyck Path



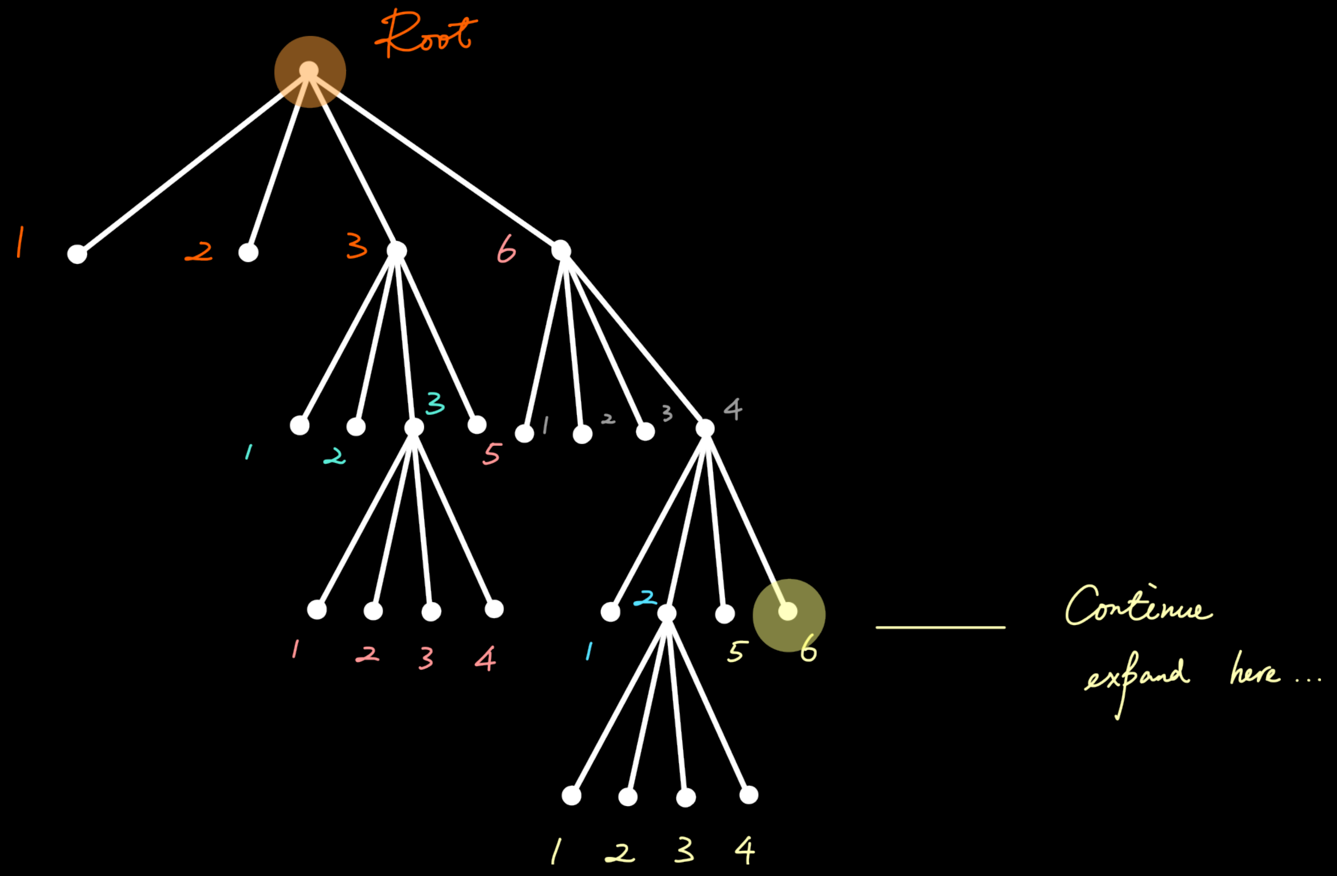
Bijection 2.0

Proper (k+1)-nary tree ⟷ Generalised (kth) Dyck Path



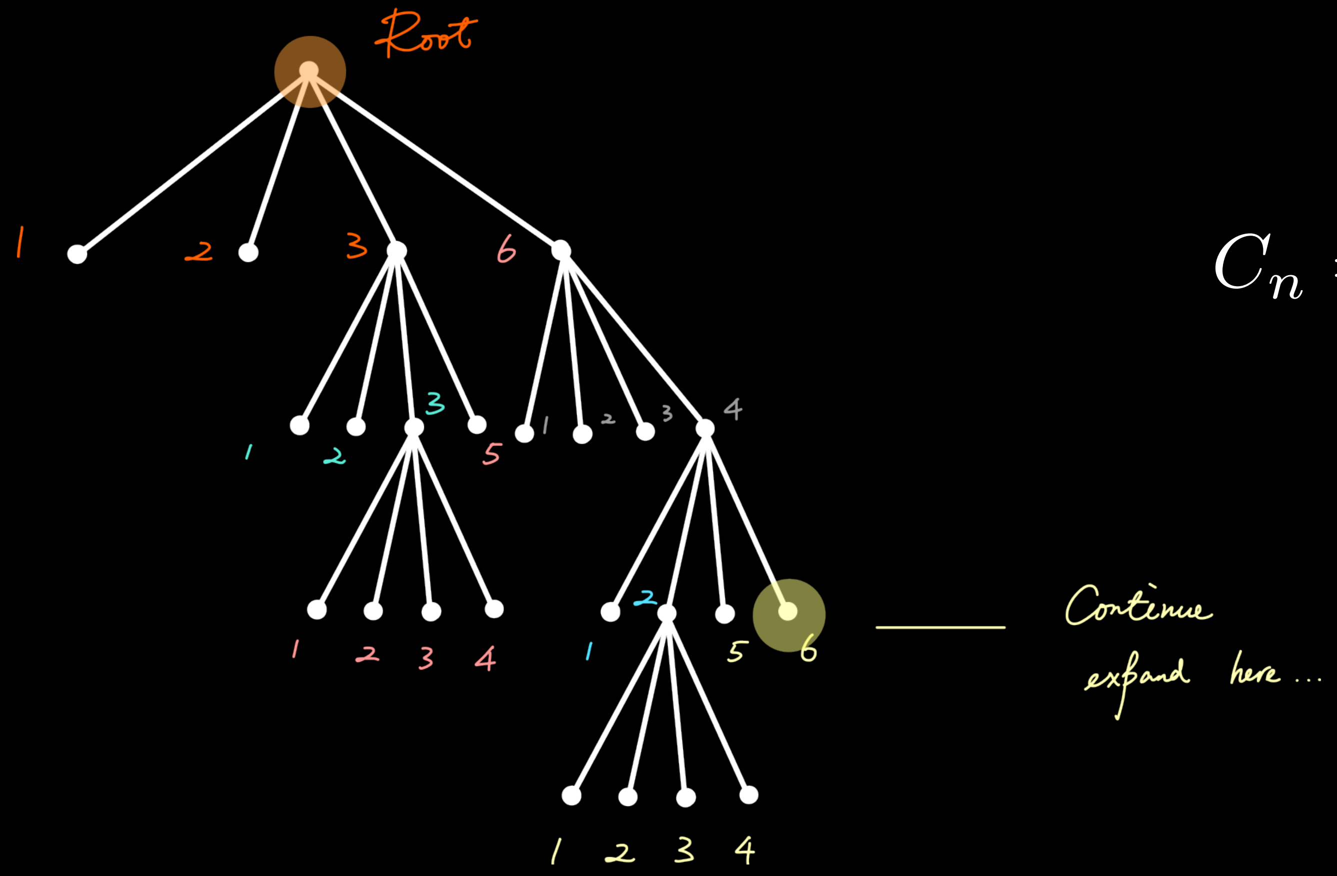
Bijection 2.0

Proper (k+1)-nary tree ⟷ Generalised (kth) Dyck Path



Bijection 2.0

Proper (k+1)-nary tree $\longleftrightarrow$ Generalised (kth) Dyck Path



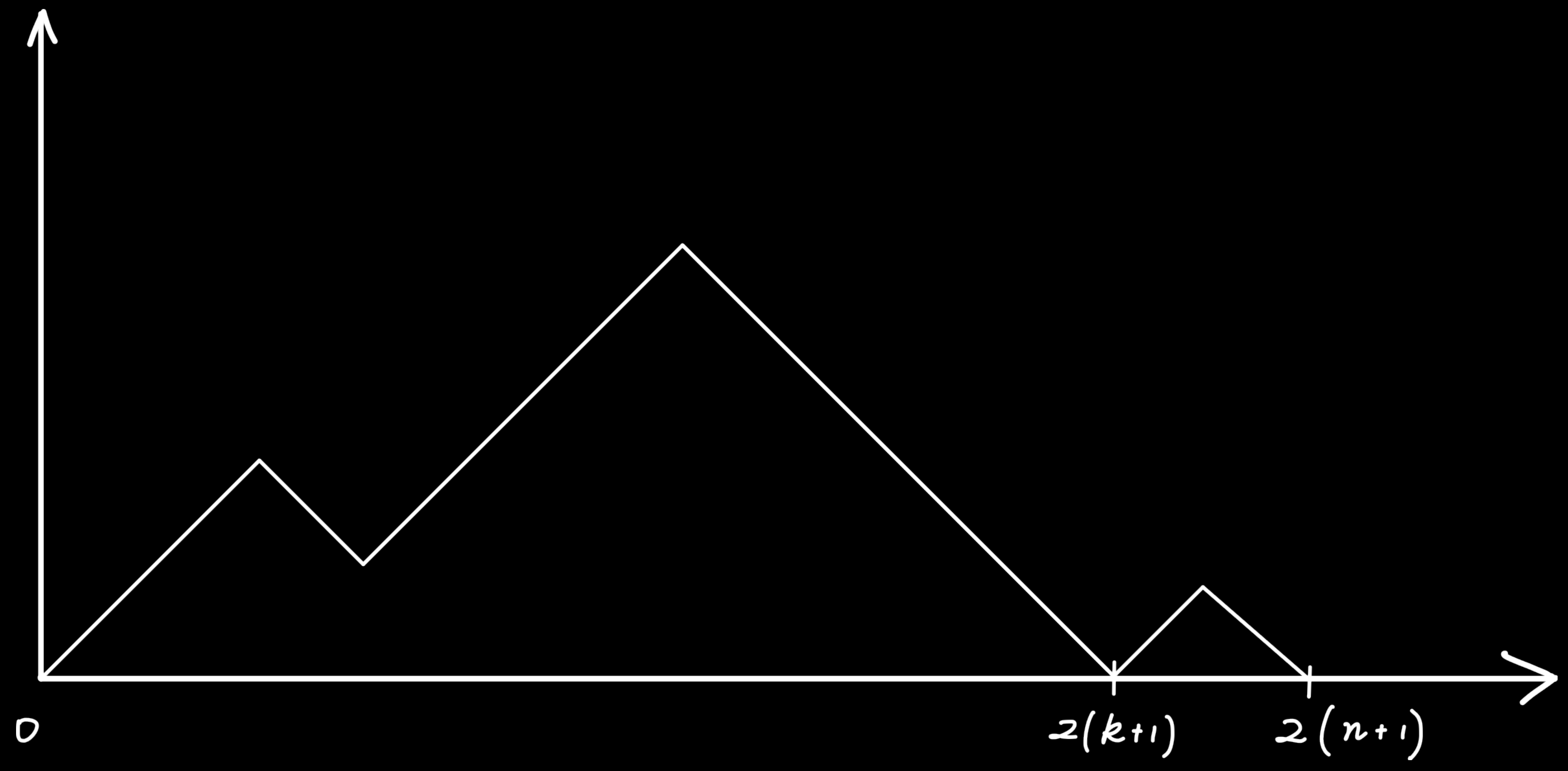
$$C_n = \frac{1}{kn+1}\binom{(k+1)n}{n}$$

Recurrence Function

$$C_{n+1} = C_0C_n + C_1C_{n-1} + ... + C_nC_0 = \sum_{k=0}^{n} C_kC_{n-k}$$

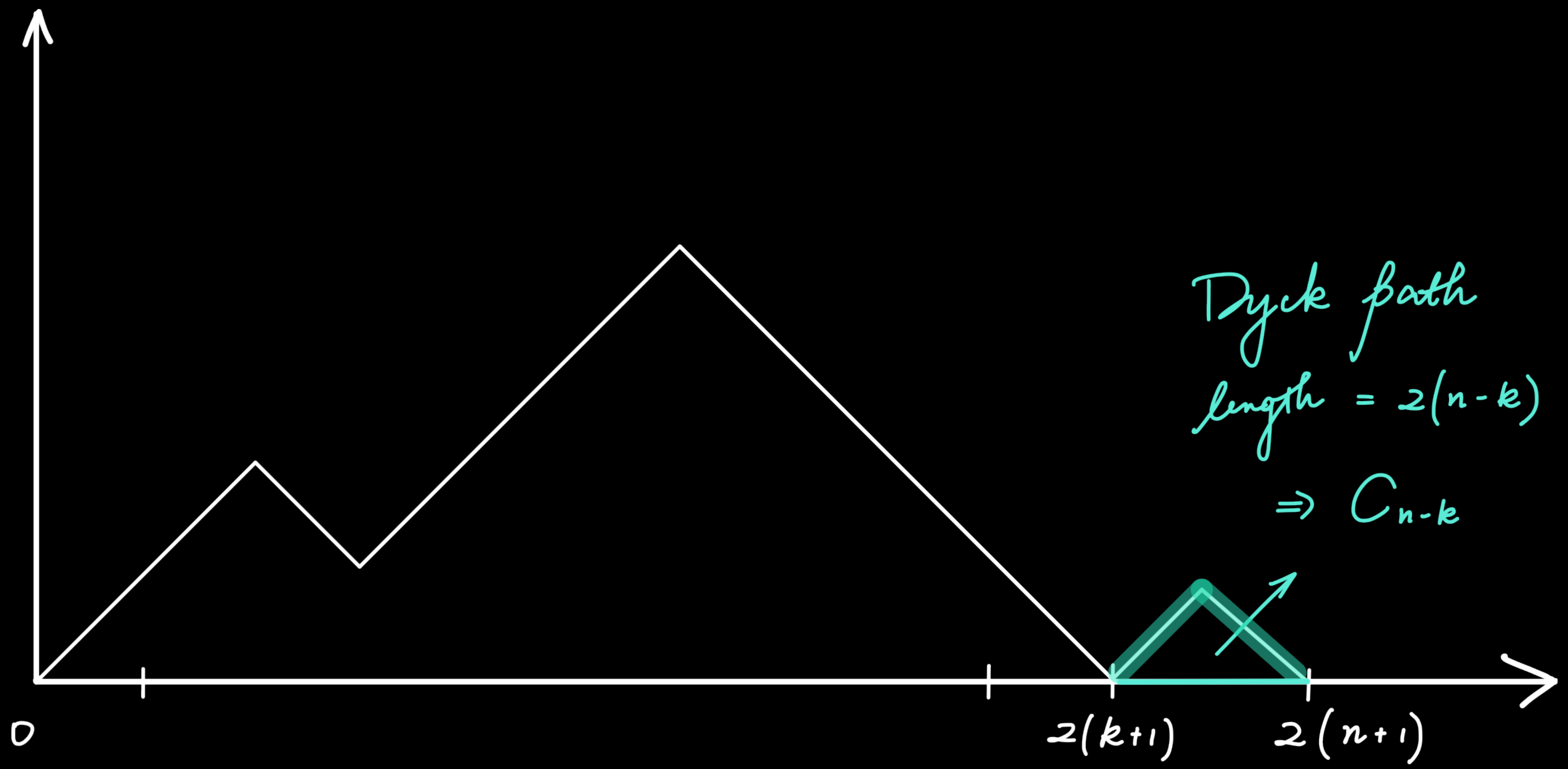
Recurrence Function

$$C_{n+1} = C_0C_n + C_1C_{n-1} + ... + C_nC_0 = \sum_{k=0}^{n} C_kC_{n-k}$$



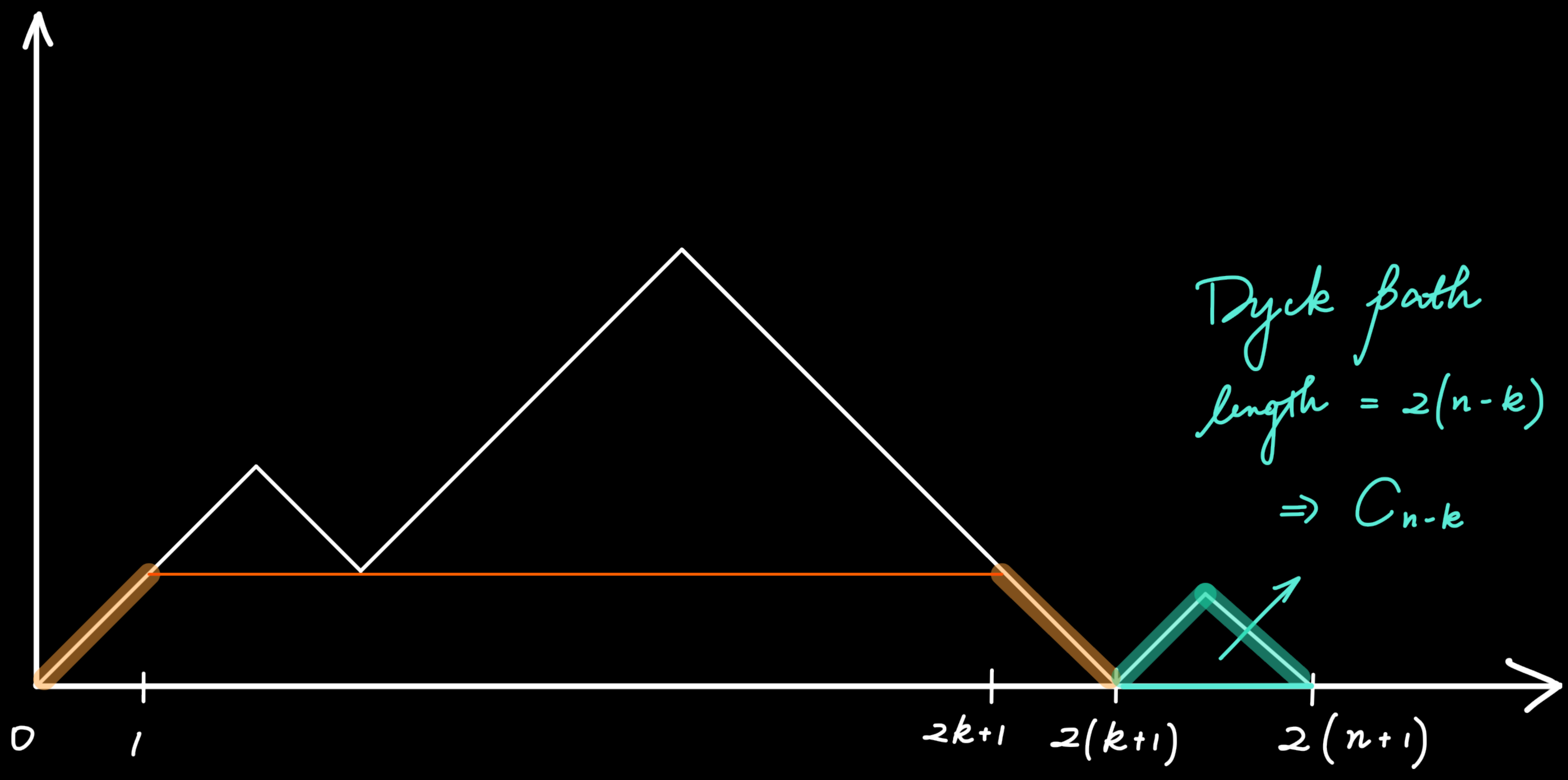
Recurrence Function

$$C_{n+1} = C_0C_n + C_1C_{n-1} + \dots + C_nC_0 = \sum_{k=0}^{n} C_kC_{n-k}$$



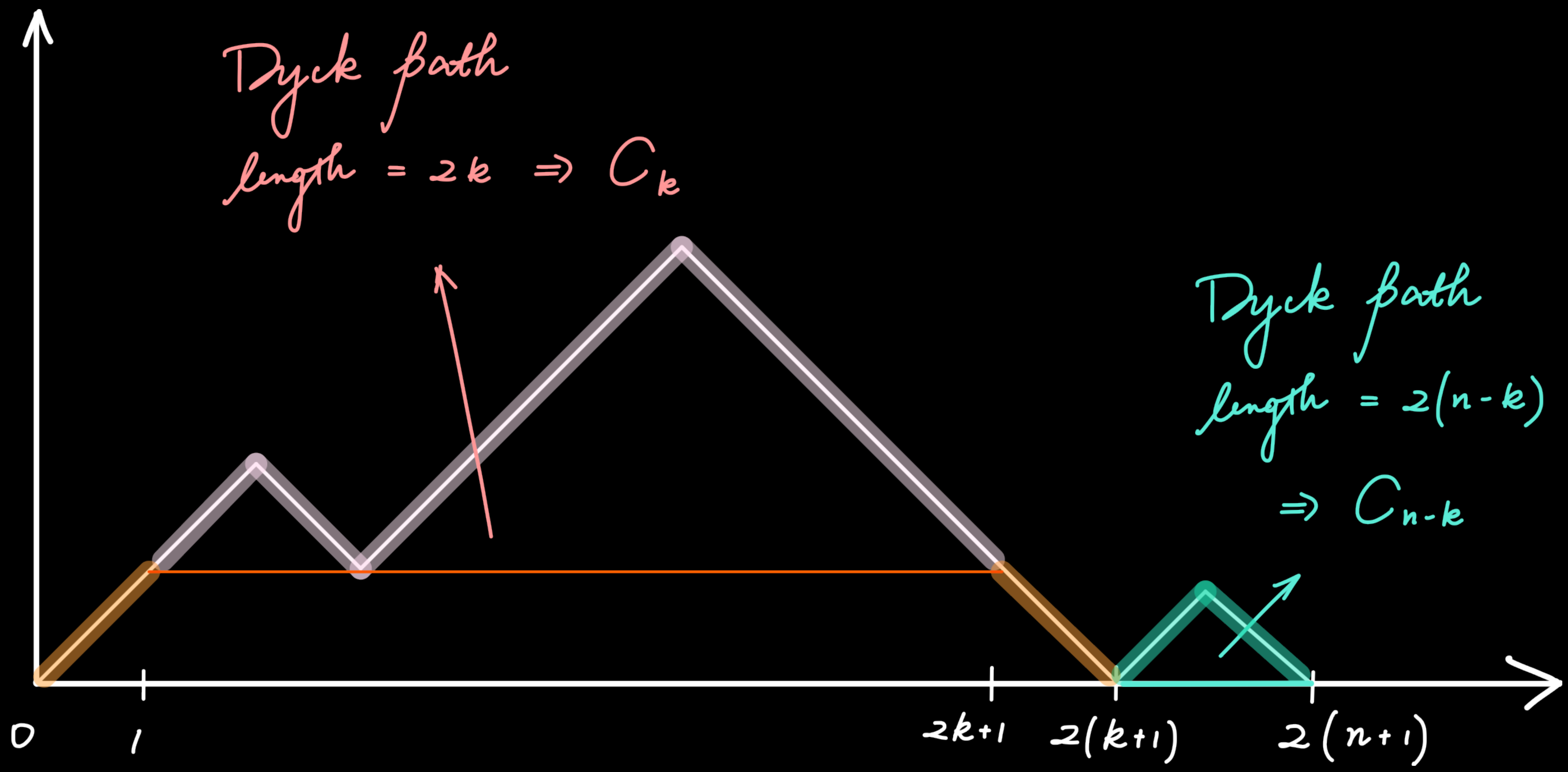
Recurrence Function

$$C_{n+1} = C_0C_n + C_1C_{n-1} + ... + C_nC_0 = \sum_{k=0}^{n} C_kC_{n-k}$$



Recurrence Function

$$C_{n+1} = C_0 C_n + C_1 C_{n-1} + \ldots + C_n C_0 = \sum_{k=0}^{n} C_k C_{n-k}$$



Recurrence Function 2.0

$$C_0 = 1, C_n = \frac{2(2n-1)}{n+1} C_{n-1}$$

Recurrence Function 2.0

$$C_0 = 1, C_n = \frac{2(2n-1)}{n+1} C_{n-1}$$

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!} \implies C_{n-1} = \frac{(2(n-1))!}{(n)!\,(n-1)!}$$

$$\frac{C_n}{C_{n-1}} = \frac{(2n)(2n-1)(2n-2)!(n-1)!}{(n+1)n(n-1)!(2n-2)!} = \frac{2(2n-1)}{n+1}$$

$$\implies C_n = \frac{2(2n-1)}{n+1} C_{n-1}.$$

Generating Function

$$f(x) = \sum_{n=0}^{\infty} C_n x^n$$

$$f(x) - 1 = \sum_{n=0}^{\infty} C_{n+1} x^{n+1}$$

$$f(x) - 1 = \sum_{n=0}^{\infty} (C_0 C_n + C_1 C_{n-1} + ... + C_n C_0) x^{n+1}$$

$$f(x) - 1 = x(f(x))^2$$

$solve\ for\ f(x):$

$$f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

Generating Function

$$f(x) = \sum_{n=0}^{\infty} C_n x^n$$

$$f(x) - 1 = \sum_{n=0}^{\infty} C_{n+1} x^{n+1}$$

$$f(x) - 1 = \sum_{n=0}^{\infty} (C_0 C_n + C_1 C_{n-1} + \ldots + C_n C_0) x^{n+1}$$

$$f(x) - 1 = x(f(x))^2$$

$$solve\ for\ f(x):$$

$$f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

$$if\ f(x) = \frac{1+\sqrt{1-4x}}{2x}$$

$$f(0) = \frac{1+1}{0},\ which\ blows\ up$$

$$Therefore,\ f(x) = \frac{1-\sqrt{1-4x}}{2x}$$

Asymptotic

$$C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$$

$$ratio \to 1\ as\ n \to \infty$$

Asymptotic

$$C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$$

$$ratio \to 1\ as\ n \to \infty$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!n!}$$

Asymptotic

$$C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$$

$$ratio \to 1\ as\ n \to \infty$$

$$Stirling's\ approx\ for\ n!$$

$$ln(n!) = n * ln(n) - n + O(ln(n))$$

$$n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!n!}$$

Asymptotic

$$C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$$

$$ratio \to 1 \; as \; n \to \infty$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!n!}$$

$$Stirling's \; approx \; for \; n!$$

$$ln(n!) = n * ln(n) - n + O(ln(n))$$

$$n! \sim \sqrt{2\pi n}(\tfrac{n}{e})^n$$

$$(n+1)! \sim \sqrt{2\pi(n+1)}(\tfrac{n+1}{e})^{n+1}$$

$$(2n)! \sim \sqrt{2\pi * 2n}(\tfrac{2n}{e})^{2n}$$

Asymptotic

$$C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$$

$$ratio \to 1 \; as \; n \to \infty$$

$$Stirling's \; approx \; for \; n!$$

$$ln(n!) = n * ln(n) - n + O(ln(n))$$

$$n! \sim \sqrt{2\pi n}(\tfrac{n}{e})^n$$

$$(n+1)! \sim \sqrt{2\pi(n+1)}(\tfrac{n+1}{e})^{n+1}$$

$$(2n)! \sim \sqrt{2\pi * 2n}(\tfrac{2n}{e})^{2n}$$

$$C_n = \tfrac{1}{n+1}\tbinom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!n!}$$

$$\sim \frac{\sqrt{4\pi n}(\frac{2n}{e})^{2n}}{\sqrt{2\pi(n+1)}(\frac{n+1}{e})^{n+1} \; \sqrt{2\pi n}(\frac{n}{e})^n}$$

$$= \frac{2^{2n}(n)^{2n}}{\sqrt{(n+1)}(n+1)^{n+1} \; \sqrt{\pi}n^n}$$

Asymptotic

$$C_n \sim \frac{4^n}{n^{3/2}\sqrt{\pi}}$$

$$ratio \to 1 \; as \; n \to \infty$$

$$Stirling's \; approx \; for \; n!$$

$$ln(n!) = n * ln(n) - n + O(ln(n))$$

$$n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$$

$$(n+1)! \sim \sqrt{2\pi(n+1)}(\frac{n+1}{e})^{n+1}$$

$$(2n)! \sim \sqrt{2\pi * 2n}(\frac{2n}{e})^{2n}$$

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$= \frac{(2n)!}{(n+1)!n!}$$

$$\sim \frac{\sqrt{4\pi n}(\frac{2n}{e})^{2n}}{\sqrt{2\pi(n+1)}(\frac{n+1}{e})^{n+1}\;\sqrt{2\pi n}(\frac{n}{e})^n}$$

$$= \frac{2^{2n}(n)^{2n}}{\sqrt{(n+1)}(n+1)^{n+1}\;\sqrt{\pi}n^n}$$

$$\because lim_{n\to\infty} n+1 = n$$

$$\sim \frac{4^n(n)^{2n}}{\sqrt{n}n^{n+1}\;\sqrt{\pi}n^n}$$

$$= \frac{4^n}{n^{3/2}\sqrt{\pi}}$$

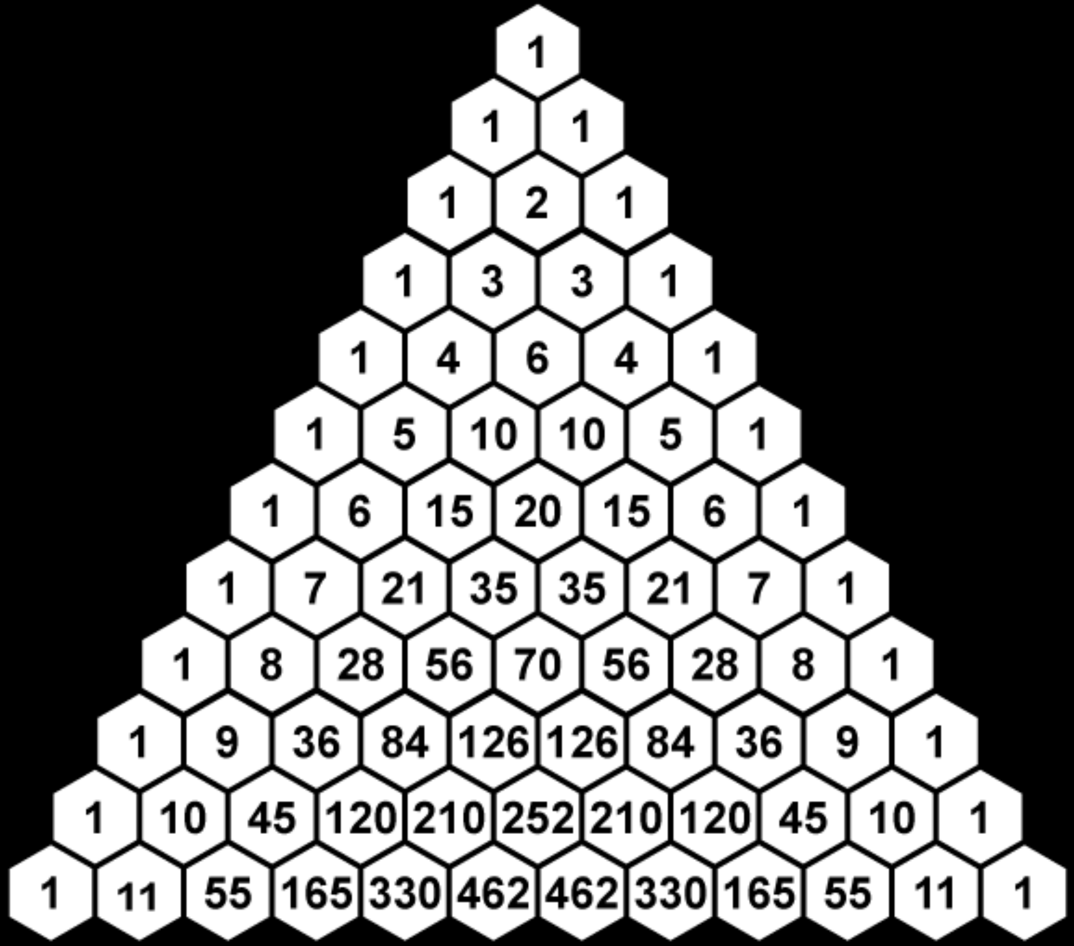
In Pascal's Triangles

1, 1, 2, 5, 14, 42, 132, 429,

1430, 4862, 16796, 58786,

208012, 742900, 2674440,

9694845...



In Pascal’s Triangles

1, 1, 2, 5, 14, 42, 132, 429,

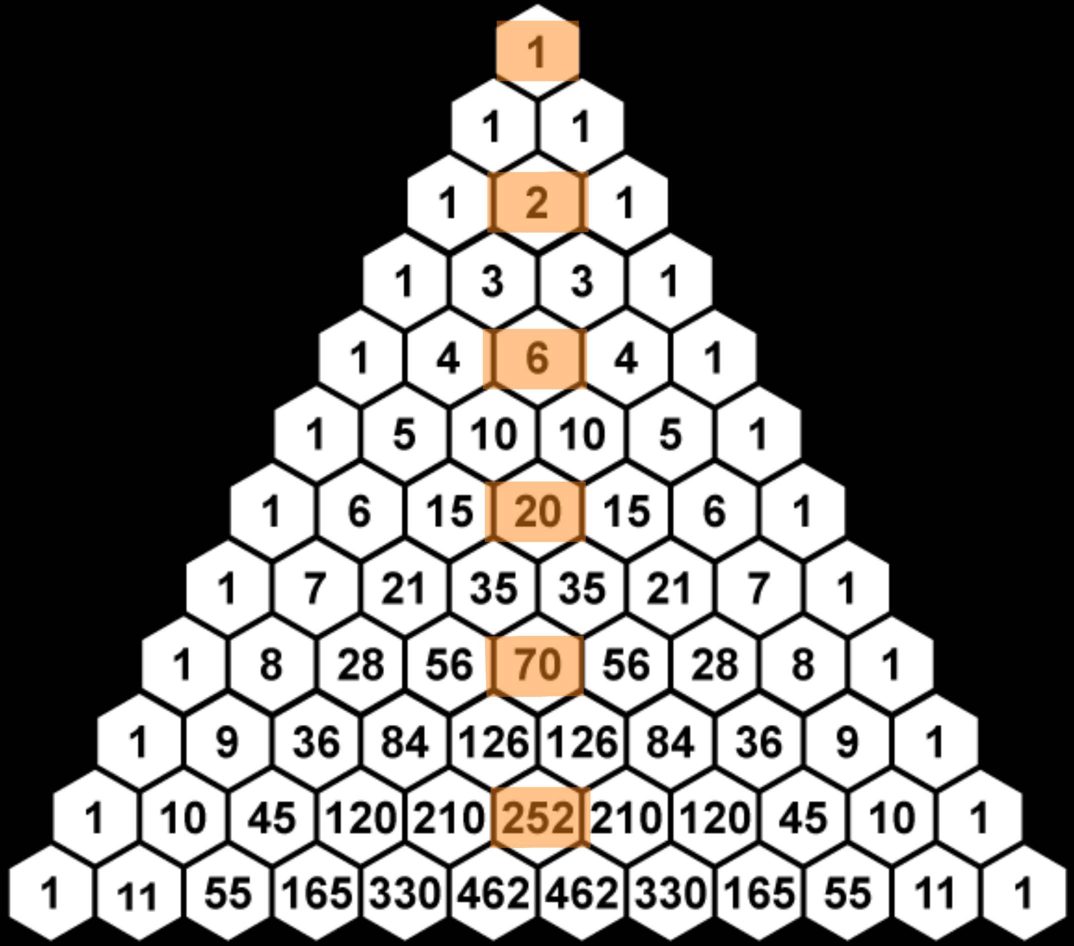
1430, 4862, 16796, 58786,

208012, 742900, 2674440,

9694845...

$Strategy\ 1$

$$C_n = \frac{n\ th\ middle\ number}{n}$$



In Pascal's Triangles

1, 1, 2, 5, 14, 42, 132, 429,

1430, 4862, 16796, 58786,

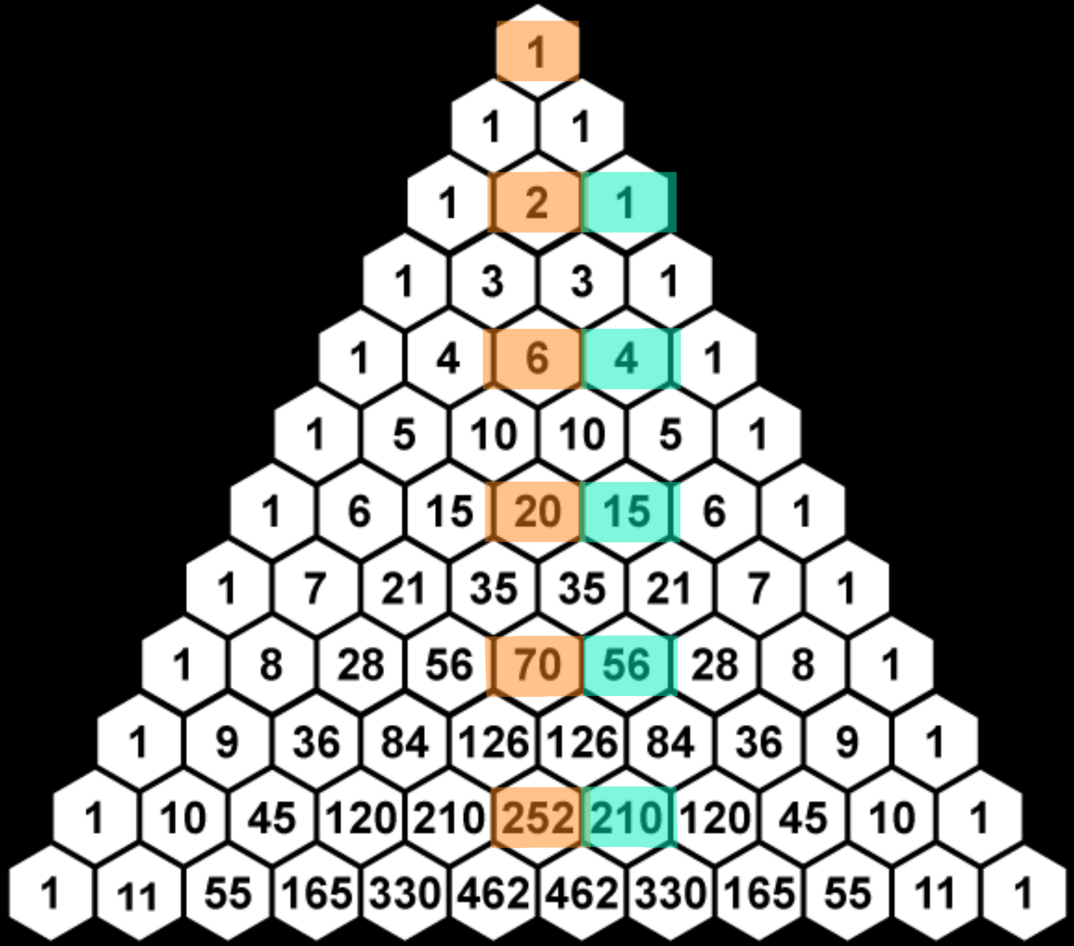
208012, 742900, 2674440,

9694845...

Strategy 2

$$C_n = n\,th\ middle\ number - the\ number\ next\ to\ it$$

$$= \binom{2n}{n} - \binom{2n}{n+1}$$



Catalan Triangle

1, 1, 2, 5, 14, 42, 132, 429,

1430, 4862, 16796, 58786,

208012, 742900, 2674440,

9694845...

$$c_{nk} = \frac{(n+k)!\,(n-k+1)}{k!\,(n+1)!}$$

$$c_{n\,n} = C_n$$

$$c_{n\,k} = c_{n-1\,k} + c_{n\,k-1}$$

1						
1	1					
1	2	2				
1	3	5	5			
1	4	9	14	14		
1	5	14	28	42	42	
1	6	20	48	90	132	132

Other Applications

Balanced parentheses

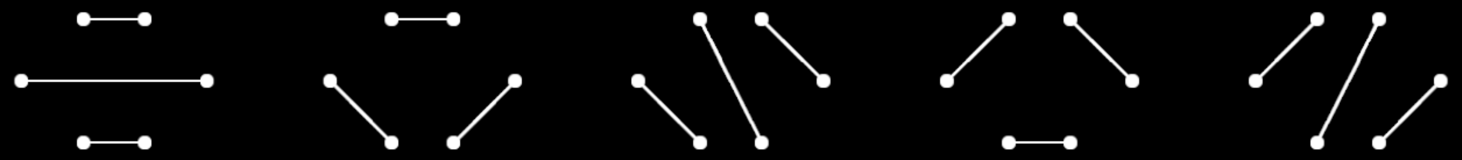
$$()(()(()))$$

Other Applications

Balanced parentheses

$()(()(()))$

n nonintersecting chords joining $2n$ points on the circumference of a circle

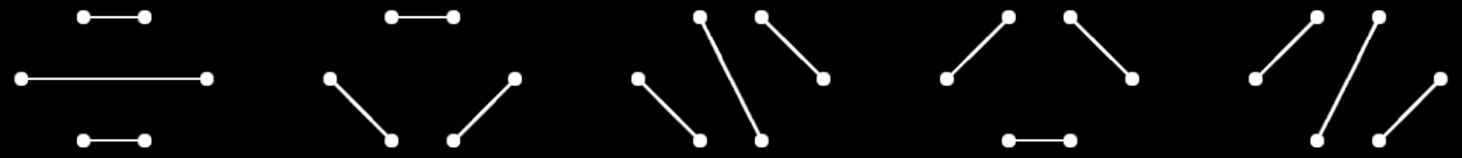


Other Applications

Balanced parentheses

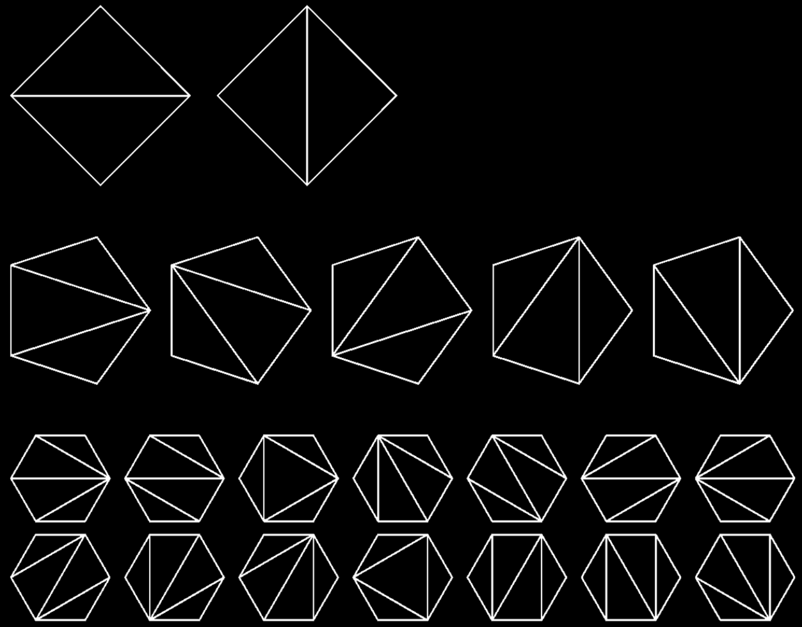
$$()(()(()))$$

n nonintersecting chords joining $2n$ points on the circumference of a circle



connecting 2n points by n non-intersecting arcs

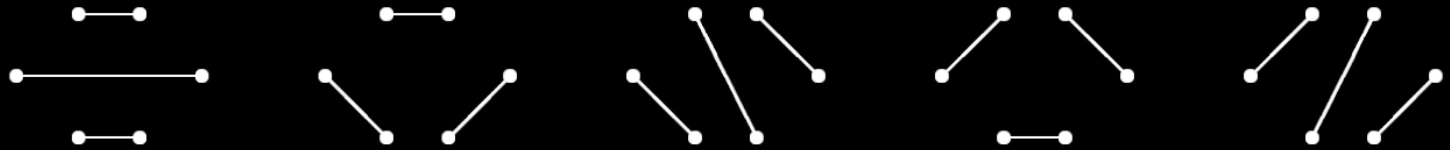
Other Applications



Balanced parentheses

()(()(()))

n nonintersecting chords joining $2n$ points on the circumference of a circle



connecting 2n points by n non-intersecting arcs

More of Just the Same

Questions: https://math.mit.edu/~rstan/ec/catalan.pdf

Solutions: https://math.mit.edu/~rstan/ec/catsol.pdf

Constructing bijections among em

Reference & Further Reading

Bijections for a class of labelled plane trees (https://doi.org/10.1016/j.ejc.2009.10.007)

Generalized Dyck Path (https://doi.org/10.1016/0012-365X(90)90039-K)

a Catalan number triangle fractal (http://www.mathrecreation.com/2009/12/catalan-number-triangle-fractal.html)

The Book of Numbers (P101-106) (http://www.blackwire.com/~bjordan/The-Book-of-Numbers.pdf)

Wikipedia/ brilliant/ Wolfram

Recursive Generation of k-ary Trees (https://cs.uwaterloo.ca/journals/JIS/VOL12/Tsikouras/tsik.pdf)

Enumerations of peaks and valleys on non-decreasing Dyck paths (https://doi.org/10.1016/j.disc.2018.06.032)

Returns and Hills on Generalized Dyck Paths (Motzkin paths)
(https://cs.uwaterloo.ca/journals/JIS/VOL19/McLeod/mcleod3.pdf)

Polygon dissections and Euler, Fuss, Kirkman and Cayley numbers (https://doi.org/10.48550/arXiv.math/9811086)

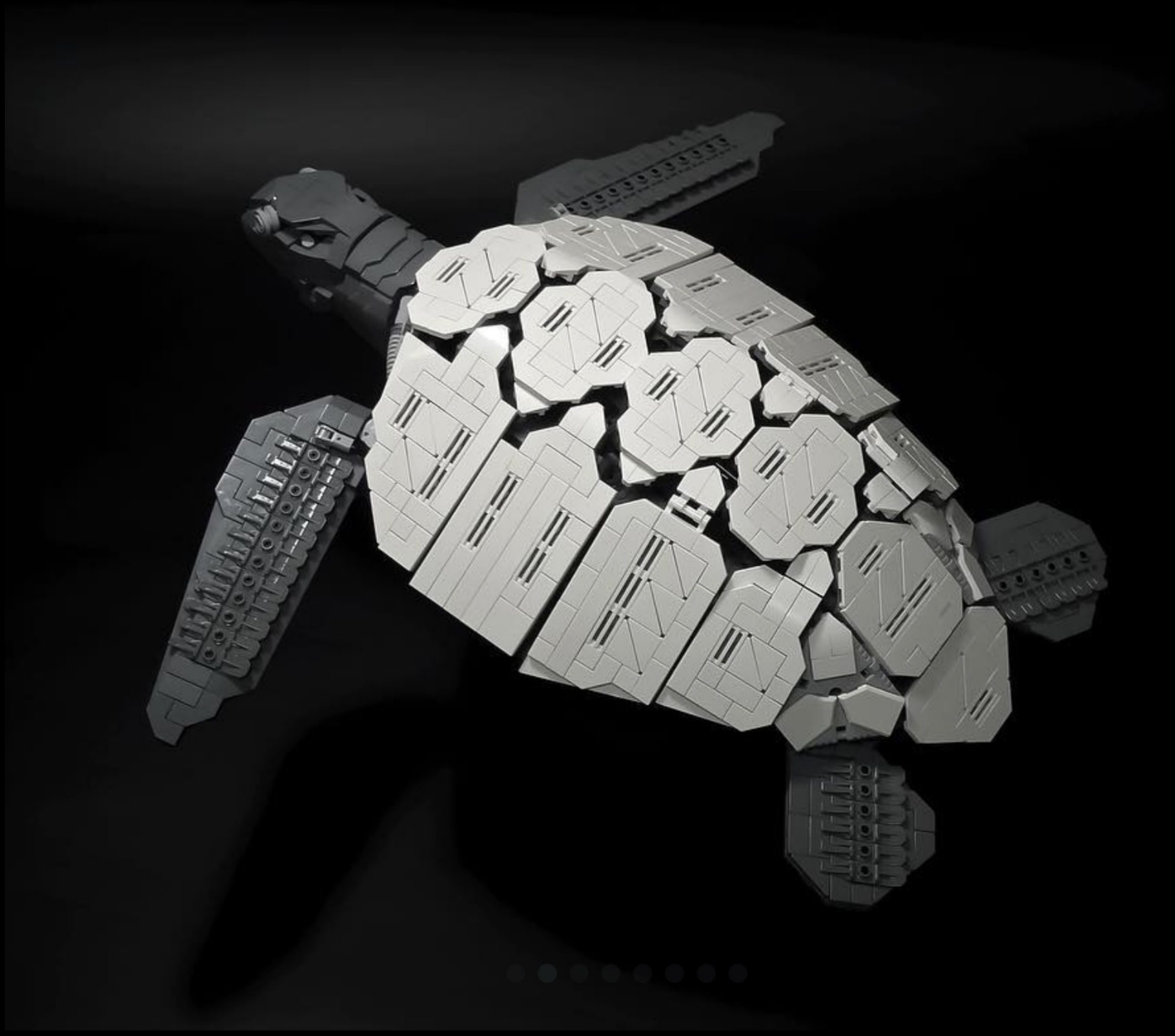
Q & A

Source of Ideas

th e
u a
r r t
t l

e :3

Source of Ideas



Stack – the example we begin with

Private tutoring MAST30012 @ Uni of Melbourne

Catalan Opening

A Turtle's Heart – Mili
https://youtu.be/6WPkgfiPeVA