Hopf algebras, braidings and quantum field theory

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What is an algebra?

Definition

An **algebra** is a vector space A equipped with a bilinear and distributive product. We call A **associative** if the product is associative, and **unital** if there exists some $I \in A$ such that $Ia = aI = a$ for all $a \in A$. Similarly, we call $u \in \ell(\mathbb{C}, A)$ a **unit** if $u(1) = I$.

An example is $\mathbb{R}^3$ with regular addition and cross-product as multiplication.

Bi-algebras and Hopf algebras

Definition

A bi-algebra is an algebra A with an additional two linear operations - a co-associative co-product $\Delta : \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$, and a co-unit $\varepsilon : A \to \mathbb{C}$.

The co-product acts by $\Delta(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$ (ie. summing a combination of tensor components, if we cannot decompose c, then we have $\Delta(c) = c \otimes I + I \otimes c$).

Definition

A Hopf algebra H is a bi-algebra that is additionally equipped with an **antipode** $S \in \ell(H)$ which is defined as,

$$\cdot(S \otimes id) \circ \Delta = \cdot(id \otimes S) \circ \Delta = u \circ \varepsilon$$

TREES! - Loday-Ronco Hopf algebra

We consider the space of planar rooted binary trees. We define the multiplication, $*$, inductively on this algebra using a dendriform structure.

$$Y \prec | = Y, \ | \prec Y = 0, \ Y \succ | = 0, \ 1 \succ Y = Y$$

$$Y \prec Y' = Y_1 \vee (Y_2 * Y') \quad Y \succ Y' = (Y * Y'_1) \vee Y'_2$$

$$* := \prec + \succ$$

We also define the antipode inductively, to make this a Hopf algebra,

$$S(|) = |, \qquad S(T) = -T - \sum_{c \in Adm(T)} S(P^c(T)) * R^c(T)$$

A Forest of Trees - the Foissy-Holtkamp Hopf algebra

This is a related Hopf algebra on the space of rooted (not necessarily binary) trees, where T/F is the tree obtained from T after removing a subforest F and all edges connecting F to the rest of T.

$$S(T) = -T - \sum_{F \in \mathcal{F}_T} S(F) \cdot (T/F)$$

The grafting operator is defined as B^+ on this algebra.

Braiding

A braiding on a vector space, V, is a function, σ, on $V \otimes V$ that satisfies the Yang-Baxter Equation,

$$(\sigma \otimes id)(id \otimes \sigma)(\sigma \otimes id) = (id \otimes \sigma)(\sigma \otimes id)(id \otimes \sigma)$$

The simplest non-trivial example is the flip operator, $\tau(a \otimes b) = b \otimes a$.

We can apply a braiding to an algebra of trees over a vector space by labelling the vertices then applying the braid operator.

Braided isomorphisms

We can define an isomorphism, Θ_σ, between these two Hopf algebras on different rooted trees by the following:

$$\Theta_\sigma(|) = 1, \qquad \Theta_\sigma(| \vee_v Y) = (B_v^+ \Theta_\sigma)(Y)$$

It can be also shown that every planar binary tree can be generated by the space $| \vee Y$, so this isomorphism is defined on the whole space. The

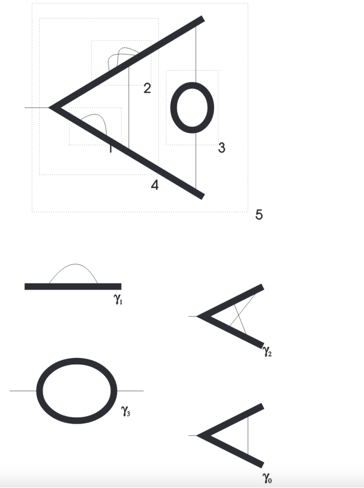
extension to the braided case was a recent (2021) result, and can be applied to more general QFT schemes.

Quantum Field Theory

- Feynman diagrams depict particle interactions.
- These can have divergences, which create infinite energy terms in calculation.
- To avoid this, we *renormalise* the Feynman diagram to contain the divergences in a finite term.
- This is a difficult or ad hoc procedure.

Enter Hopf algebras

We define a Hopf algebra structure on Feynman diagrams. This graph is of class $(((\gamma_1)(\gamma_2)\gamma_0)(\gamma_3)\gamma_0)$.



Hopf algebra structure and QFT

The Hopf algebra structure allows us to classify graphs in terms of their subdivergences. Then,

- We have a renormalisation scheme acting on subdivergences, denoted $R(x)$
- The forest formula "collects together" all counter-terms,

$$Z(\Gamma) = -R(\Gamma) - \sum_{\gamma \in \Gamma} R(Z(\gamma)\Gamma/\gamma)$$

- Note the similarity to the antipode formula before, $Z(\Gamma) = S(R(X))$.
- This gives a logical process and structure to renormalise Feynman diagrams.

References

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