Towers of Hanoi, Gray Codes, and Coxeter Groups

James Dann

May 5, 2022

Refresher: Induction

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- Prove case is true for $n = 0$
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- True for all $n \in \mathbb{N}$

Problem 1: Towers of Hanoi

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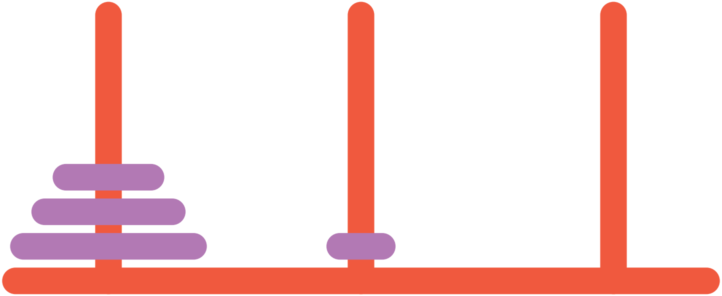

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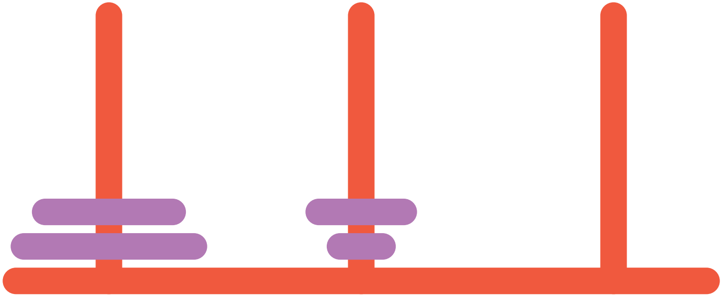
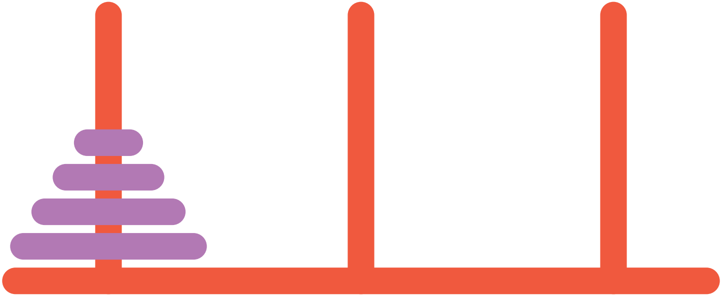


Solution 1: Towers of Hanoi



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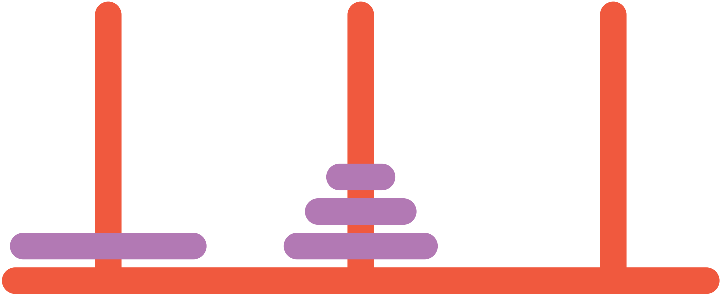

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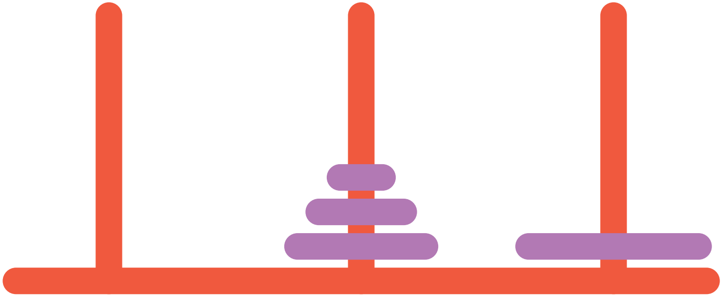

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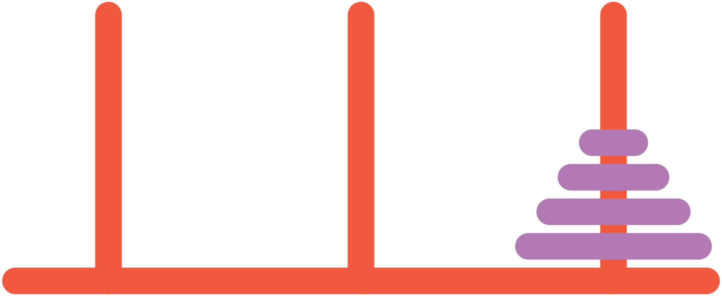

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- Connection to n-dimensional cubes

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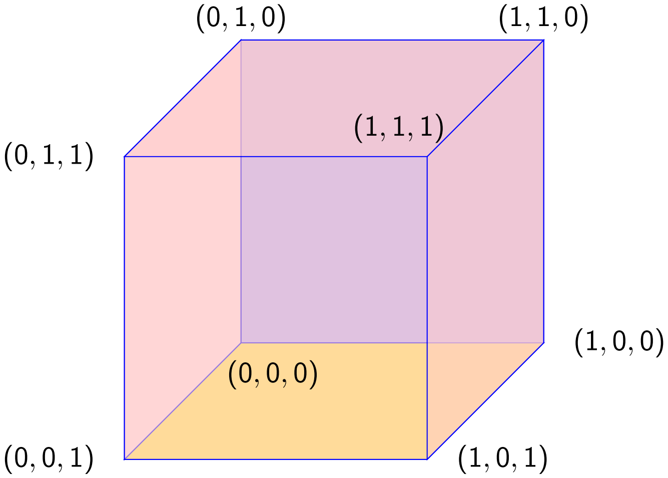
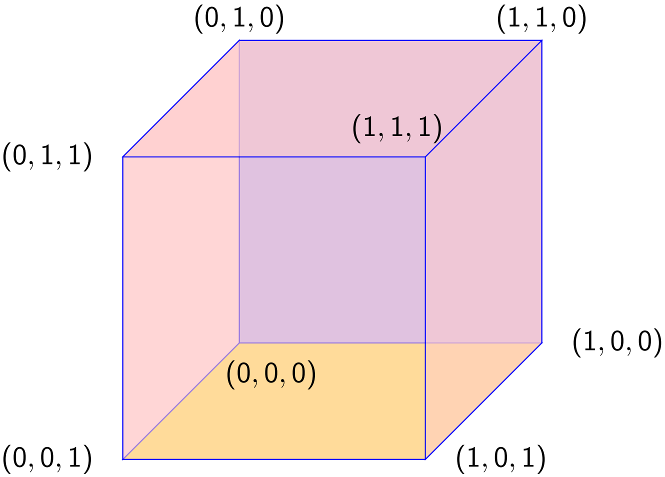
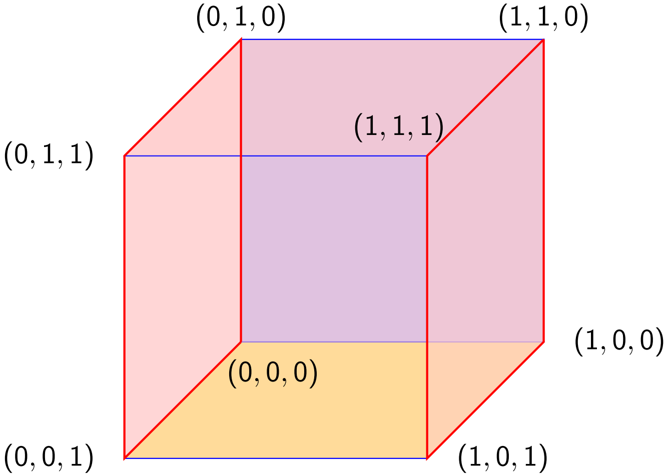


n-dimensional cubes



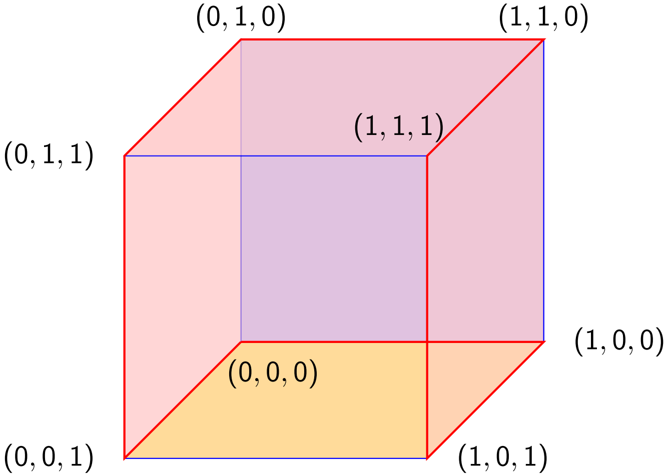
00
01
11
10

n-dimensional cubes



00
01
11
10

n-dimensional cubes



00	000
01	001
11	011
10	010
	110
	111
	101
	100

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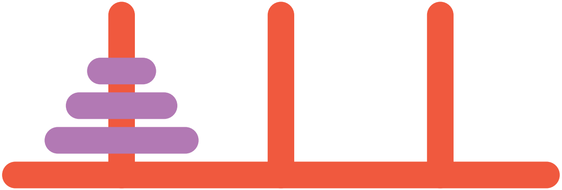
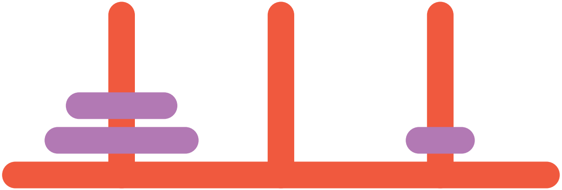
Connection??

String	Diff
000	-
001	0
011	1
010	0
110	2
111	0
101	1
100	0



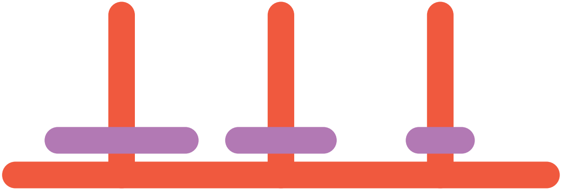
Connection??

String	Diff
000	-
001	**0**
011	1
010	0
110	2
111	0
101	1
100	0



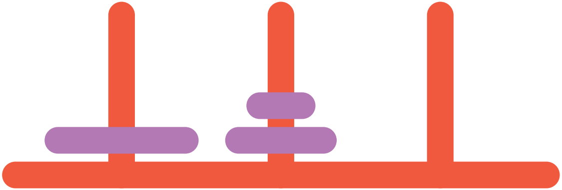
Connection??

String	Diff
000	-
001	0
011	**1**
010	0
110	2
111	0
101	1
100	0



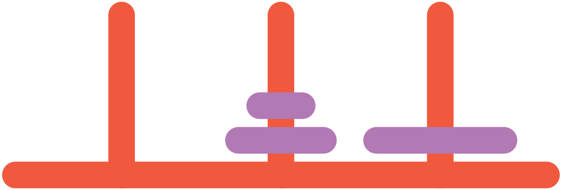
Connection??

String	Diff
000	-
001	0
011	1
010	**0**
110	2
111	0
101	1
100	0



Connection??

String	Diff
000	-
001	0
011	1
010	0
110	**2**
111	0
101	1
100	0



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001	0
011	1
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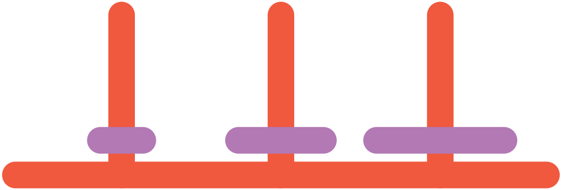

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000	-
001	0
011	1
010	0
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111	0
101	**1**
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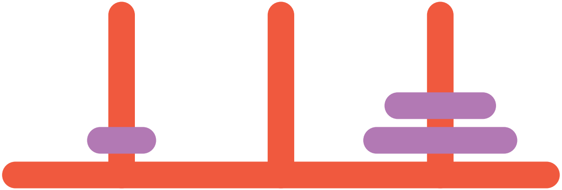

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000	-
001	0
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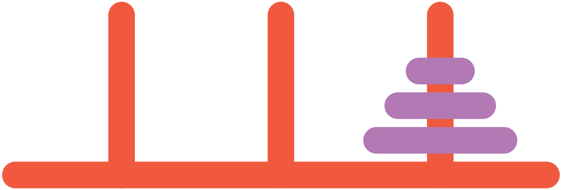

Connection??

String	Diff	Peg 1	Peg 2	Peg 3
000	-	2 1 0		
001	0	2 1		0
011	1	2	1	0
010	0	2	1 0	
110	2		1 0	2
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- Flip 0th bit $=$ swap all disks ≤ 0 between pegs containing 0 and 1, if same peg then swap with ? peg

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000	-	2 1 0		
001	0	2 1		0
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111	0	0	1	2
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100	0			2 1 0

- Flip 0th bit = swap all disks ≤ 0 between pegs containing 0 and 1, if same peg then swap with ? peg
- Flip 1st bit = swap all disks ≤ 1 between pegs containing 1 and 2, if same peg then swap with ? peg

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String	Diff	Peg 1	Peg 2	Peg 3
000	-	2 1 0		
001	0	2 1		0
011	1	2	1	0
010	0	2	1 0	
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- Flip 2nd bit = swap all disks ≤ 2 between left peg and peg with 2, if same peg then swap with ? peg

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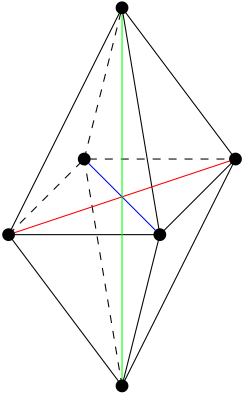
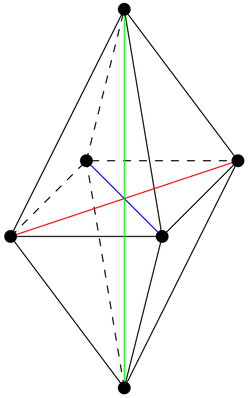


A generalisation

- Think about the set of all binary strings W as a group
- $\mathbb{Z}_2^n$
- Reflections
- Hang on if you don't know group theory...



Reflection Group

- Three "simple" generators s_0, s_1, s_2 such that $1 = s_0^2 = s_1^2 = s_2^2$

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$$\mathbb{Z}_2^n = \left\langle s_0, s_1, s_2 \mid 1 = s_i^2 = (s_0 s_1)^2 = (s_1 s_2)^2 = (s_0 s_2)^2 \right\rangle$$

Coxeter System

- Set of generators $S = \{s_1, s_2, \ldots, s_n\}$

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- Represents reflections in (hyper)planes
- Every generator is its own inverse ($s_i^2 = 1$)
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- Represents angle between s_i and s_j: $\frac{\pi}{m(i,j)}$

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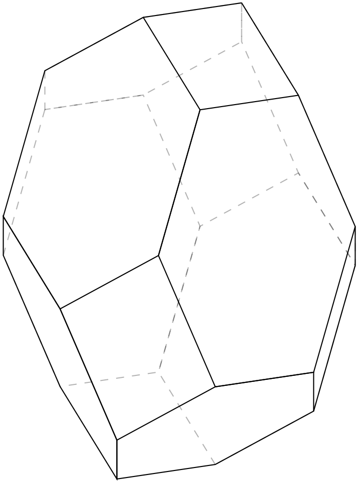
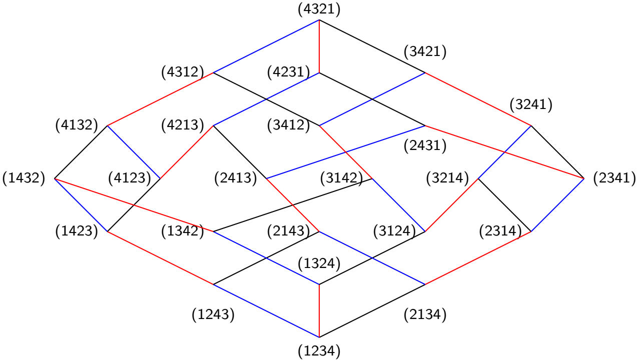
- Satisfy aforementioned relations, with $(s_i s_{i+1})^3 = 1$

Tikz is hard



Decomposition



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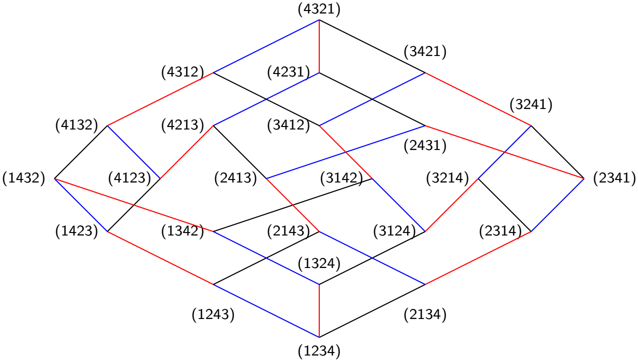
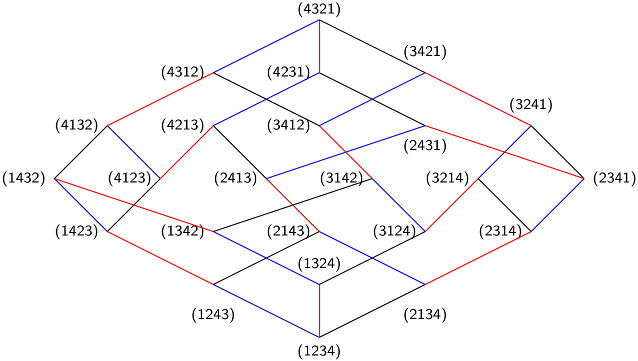


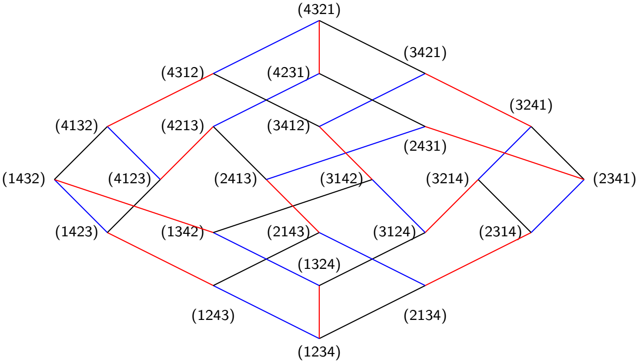
Decomposition

- Inductive decomposition
- $g =$ something that ends only in blue * something never using blue



Decomposition

- Split into smaller parts with Hamilton cycles



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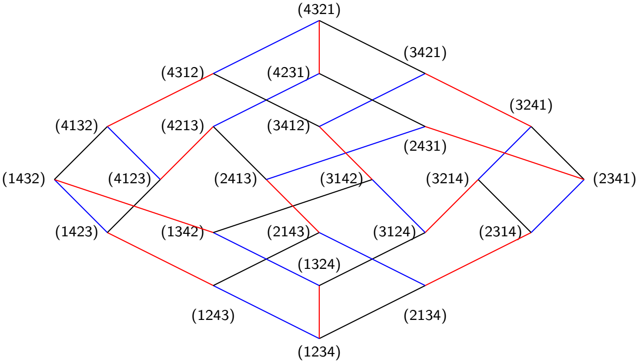

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