An Unexpected Equivalence From A Silly Cross Product Puzzle

Max Orchard

August 26, 2022

Inspiration

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A paper by Kauffman:

JOURNAL OF COMBINATORIAL THEORY, Series B **48**, 145–154 (1990)

[SPOILER] and the Vector Cross Product*

LOUIS H. KAUFFMAN

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Communicated by the Managing Editors

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[SPOILER] is equivalent to a combinatorial problem about the three-dimensional vector cross product algebra.

The Puzzle

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The *cross product* is defined on this basis by

$$i \times i = j \times j = k \times k = 0,$$
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$$j \times i = -k, \quad k \times j = -i, \quad i \times k = -j,$$

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We will only be looking at the cross product on $\{\pm i, \pm j, \pm k, 0\}$.

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Off-topic remark: the number of associations of n variables is equal to the n^{th} Catalan number. The sequence of Catalan numbers is

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots$$

They grow fairly quickly!

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Find *sharp* solutions to the equation $L(X_1, \ldots, X_n) = R(X_1, \ldots, X_n)$.

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$X_1 = i$, $X_2 = k$, $X_3 = i$ is a sharp solution to $L = R$, because

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The existence of a sharp solution to the equation $L = R$ for any $n \in \mathbb{Z}^+$ and for all associations L, R of the variables $X_1, \ldots, X_n$ is equivalent to

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The existence of a sharp solution to the equation $L = R$ for any $n \in \mathbb{Z}^+$ and for all associations L, R of the variables $X_1, \ldots, X_n$ is equivalent to **the four colour theorem**.

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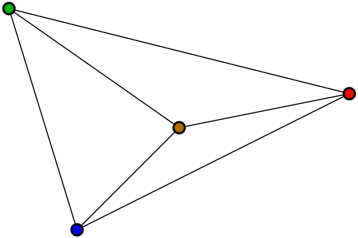

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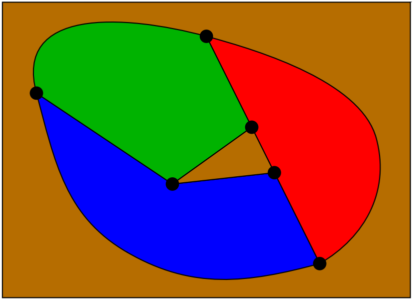

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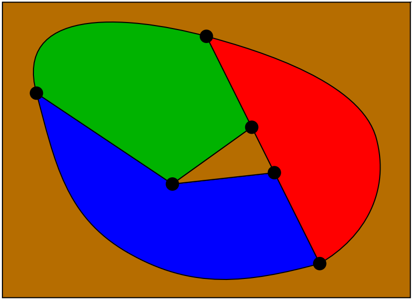
Theorem (Four Colour Theorem)

Every bridgeless cubic planar graph can be face-coloured with four colours.



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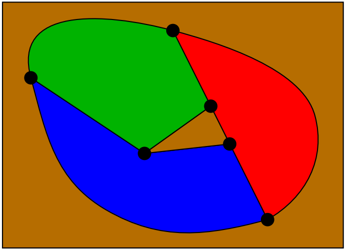

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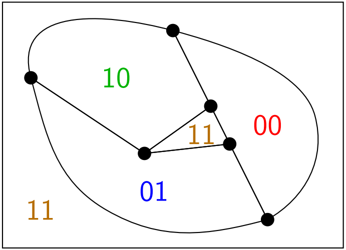

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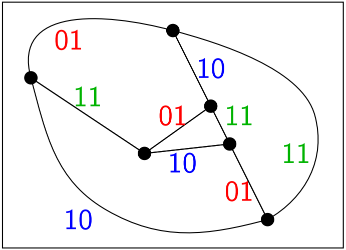

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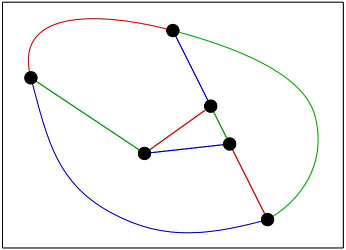

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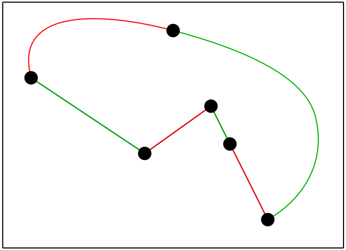

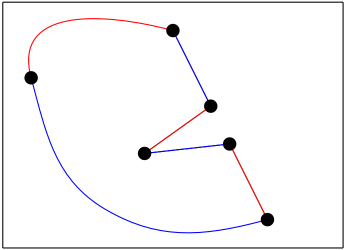

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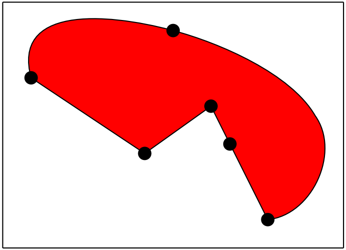

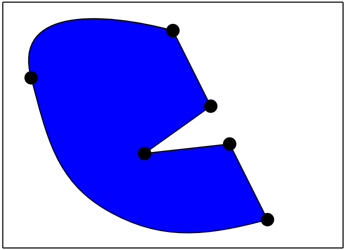

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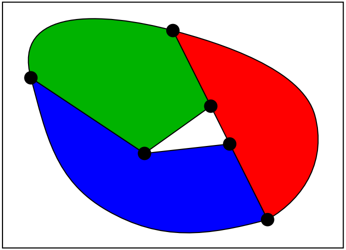

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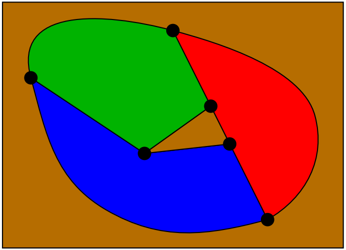

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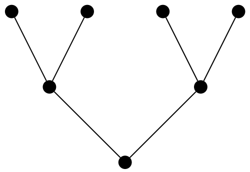

From Association To Graph

Now, flip the tree for R horizontally (so there is no crossover). Pair up corresponding leaves with an edge (representing that X_i in L is equal to X_i in R), and pair up the roots with an edge (as we want $L = R$).

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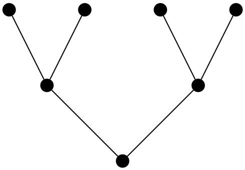

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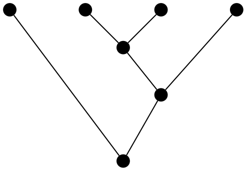

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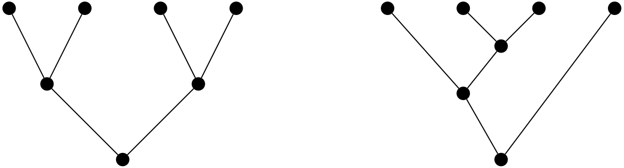

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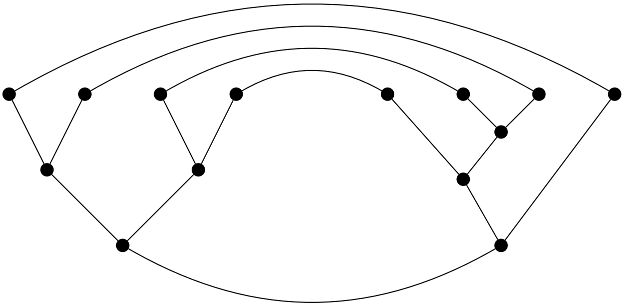

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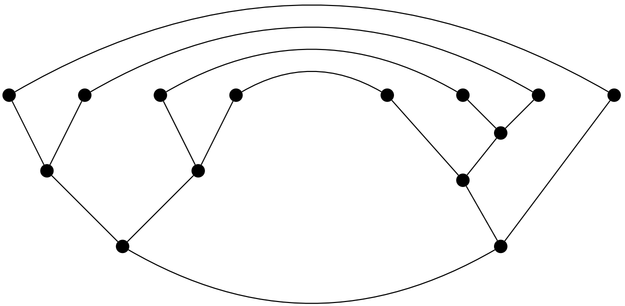

By removing the leaf vertices, this forms a bridgeless cubic planar graph.

Sharp Solution $\implies$ Four Colour Theorem

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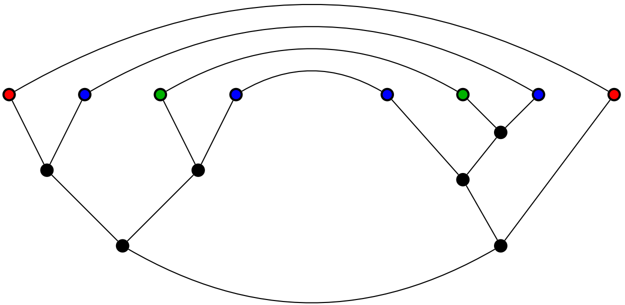

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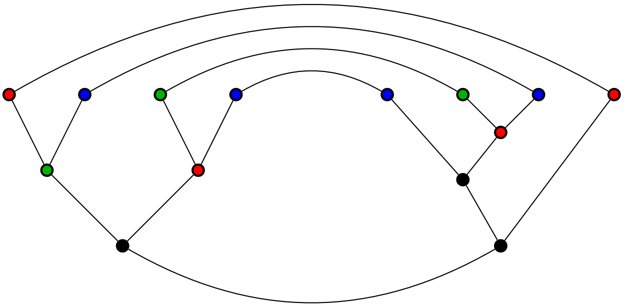

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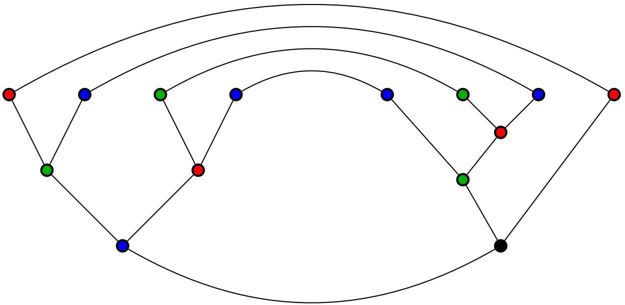

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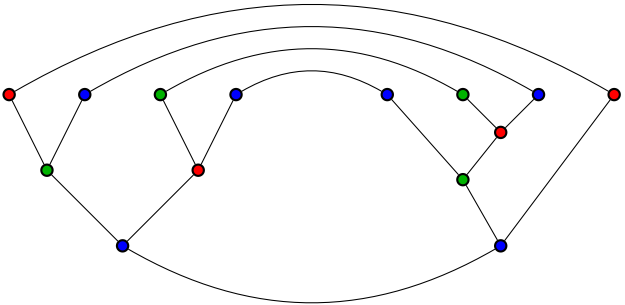

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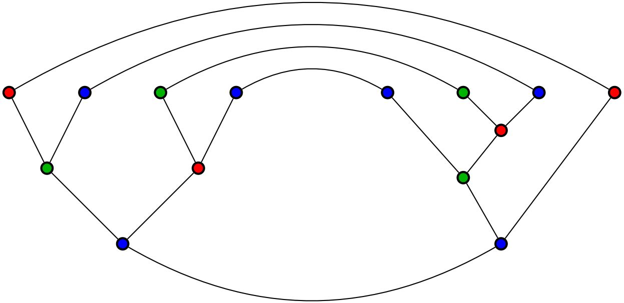

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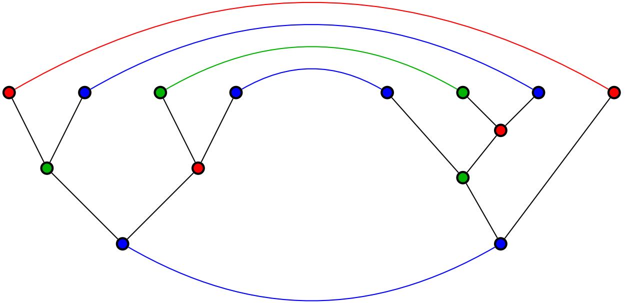

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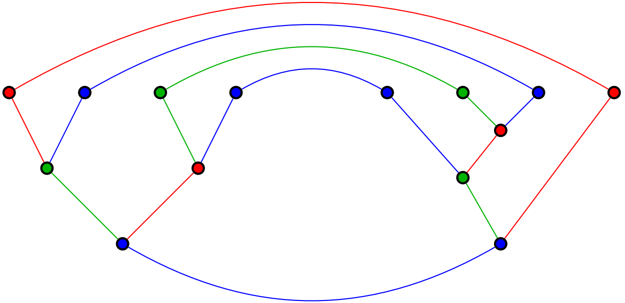

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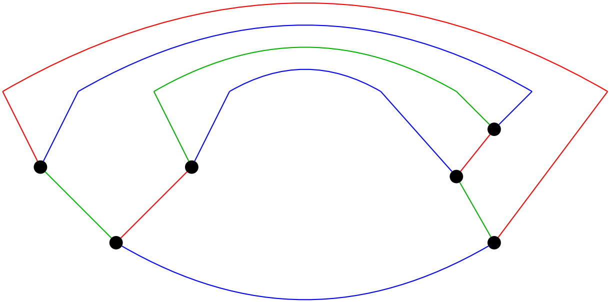

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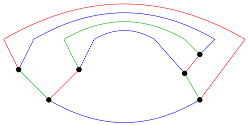

Sharp Solution $\Longrightarrow$ Four Colour Theorem

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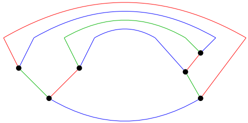
Colour the graph as before. This is a 3-colouring as we have a sharp solution (so the only possible options for the vertices are $\{\pm i, \pm j, \pm k\}$) and we are ignoring signs.



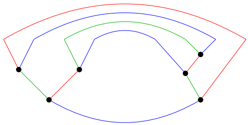
Sharp Solution $\implies$ Four Colour Theorem

Theorem

Given a sharp solution to $L = R$, we can obtain a Tait colouring of the associated bridgeless cubic planar graph.

Proof.

Colour the graph as before. This is a 3-colouring as we have a sharp solution (so the only possible options for the vertices are $\{\pm i, \pm j, \pm k\}$) and we are ignoring signs. It is a proper colouring due to the cyclic nature of the cross product on $\{i, j, k\}$ (ignoring signs). □



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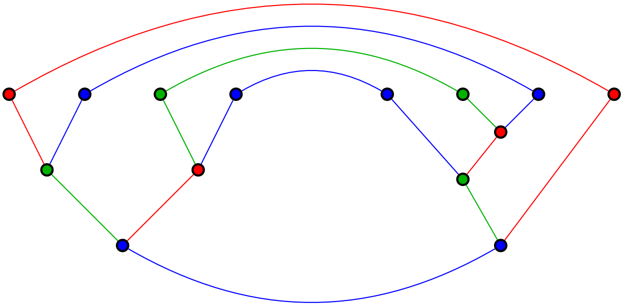

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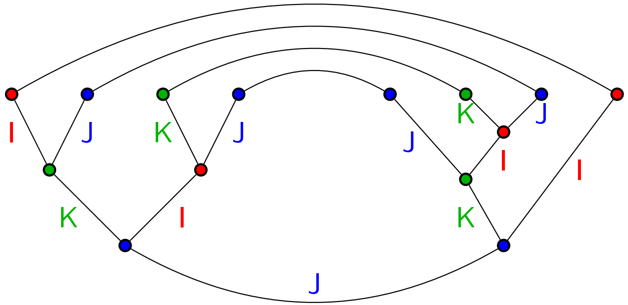

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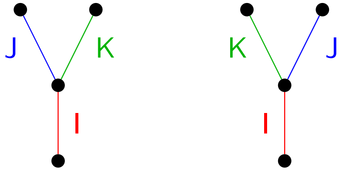

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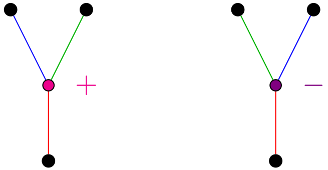

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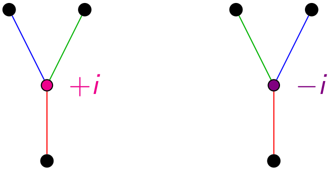

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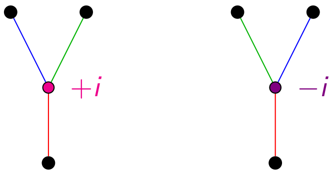

Because of this assignment, multiplying the labels for L's tree will "give" the sign of $L(X_1, \ldots, X_n)$, and similarly for R's tree. This follows from bilinearity.

Four Colour Theorem $\implies$ Sharp Solution

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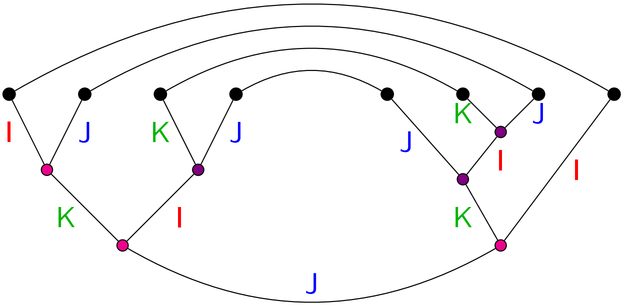

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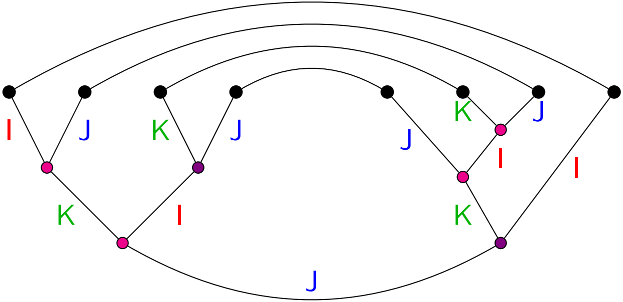

As the tree for R is flipped, we must flip the labelling on the right.

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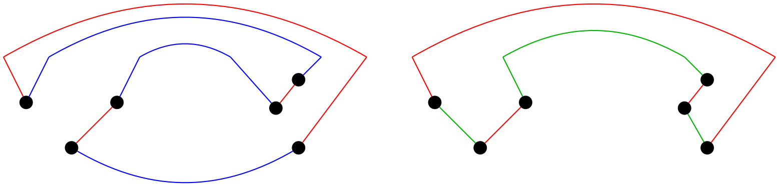

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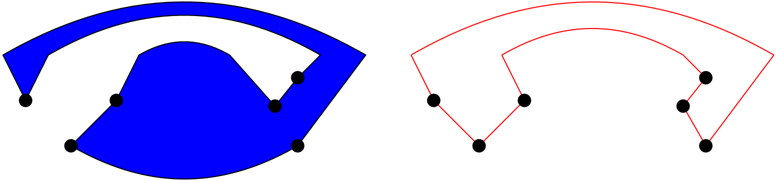

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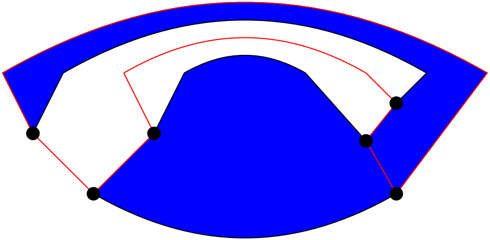

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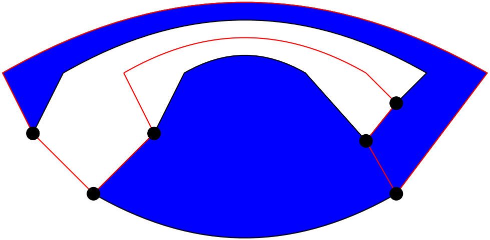
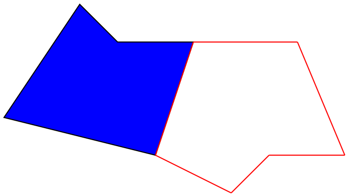


When two formations overlap, there are two different ways their edges can interact.

Formations

A *bounce* occurs when both formations share an edge and the interior of each formation is entirely disjoint, or one is contained inside the other.



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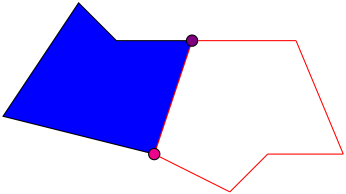
Each bounce consists of a $+i$ vertex and a $-i$ vertex. Therefore, each bounce contributes 1 to the complex product of the labels.



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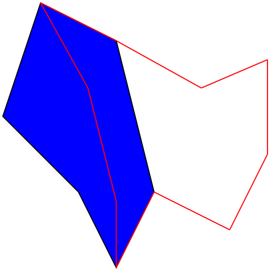

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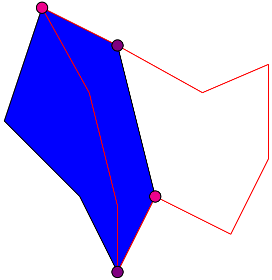

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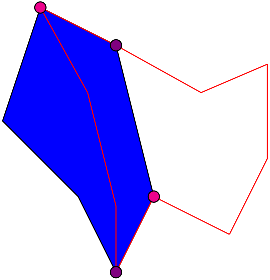

Lemma

The complex product of the labels in a Tait colouring is 1.

Four Colour Theorem $\implies$ Sharp Solution

Theorem

Given a Tait colouring of the associated bridgeless cubic planar graph, we can obtain a sharp solution to $L = R$.

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Thus $e = e'$, and we have a sharp solution to $L = R$. □

References

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