Weyl's law: When analysis informs geometry

James Stanfield

October 22, 2021

Drums and the wave equation

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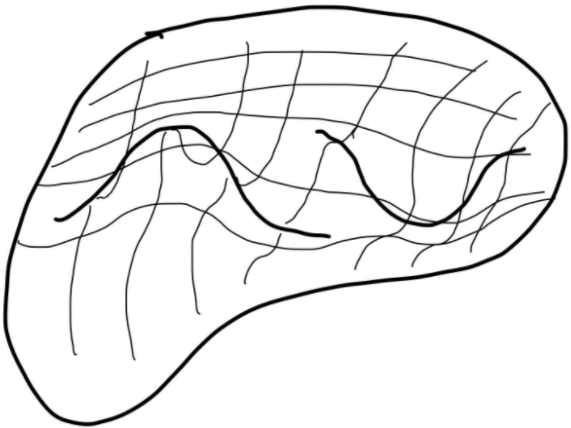

Drums and the wave equation

Let $\Omega \subset \mathbb{R}^2$ be a "drum". Suppose $u\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is the height of the drumhead over time (written $u(x,t)$). Then u (approximately) satisfies the *wave equation*

$$\frac{\partial^2 u}{\partial t^2} = \Delta u. \tag{1}$$

Here, $\Delta u := \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$. The drumhead is clamped at the boundary, so that $u(x,t) = 0$ for all $x \in \partial\Omega$ and $t \in \mathbb{R}$.

Wave equation: Standing wave solutions

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Consider *standing wave solutions* of the form

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For $\lambda > 0$ and $v : \Omega \to \mathbb{R}$. Clearly $v|_{\partial\Omega} = 0$.

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$$0 = \frac{\partial^2 u}{\partial t^2}(x,t) - \Delta u(x,t) = -\lambda \cos(\sqrt{\lambda} t) v(x) - \cos(\sqrt{\lambda} t) \Delta v(x) \iff \Delta v + \lambda v = 0.$$

So $v\colon \Omega \to \mathbb{R}$ is required to satisfy $\Delta v + \lambda v = 0$ (look familiar?).

Studying the eigenvalue problem $-\Delta v = \lambda v$ (the one dimensional case)

Suppose $\Omega = (0, L) \subset \mathbb{R}$ so $v\colon (0, L) \to \mathbb{R}$. $v(0) = v(L) = 0$.

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These are the *harmonics* of a string of length L. (Someone remind James to draw a picture).

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Notice we can recover L from the eigenvalues. We can *hear* the length of a string!

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Suppose $n = 2$, $\Omega = (0, L_1) \times (0, L_2)$. Then $0 = \Delta v + \lambda v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \lambda v$.

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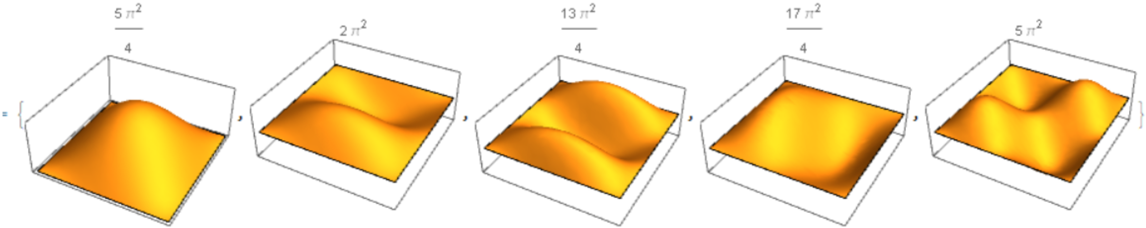

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So $N(\lambda) = \#$ of lattice points $(n, m) \in \mathbb{N} \times \mathbb{N}$ inside the ellipse $\frac{x^2}{(\sqrt{\lambda}\frac{L_1}{\pi})^2} + \frac{y^2}{(\sqrt{\lambda}\frac{L_2}{\pi})^2} = 1$

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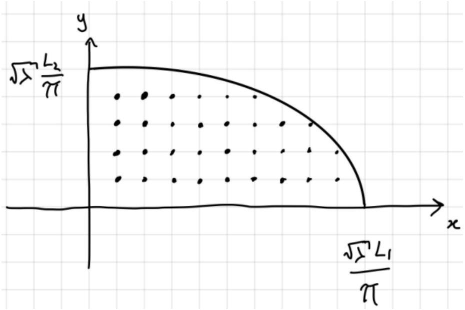

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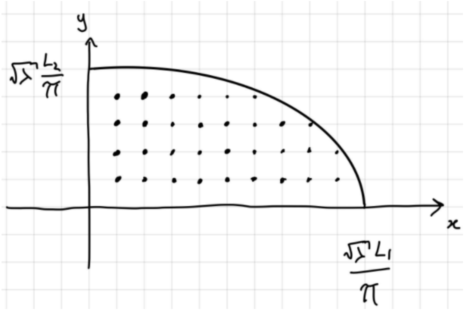

For large λ,

$$N(\lambda) \approx \frac{\text{Area(Ellipse)}}{4} = \pi\lambda\frac{L_1 L_2}{4\pi^2} = \lambda\frac{\text{Area}(\Omega)}{4\pi}.$$

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Theorem (Weyl, 1912)

For any drum $\Omega \subset \mathbb{R}^2$, $N(\lambda) = \lambda \frac{\text{Area}(\Omega)}{4\pi} + o(\lambda)$ *as* $\lambda \to \infty$.

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Upshot: We can hear the area of a drum!
Question: Can we hear more?

Conjecture (Weyl)

$N(\lambda) = \lambda \frac{\text{Area}(\Omega)}{4\pi} - \sqrt{\lambda}\frac{\text{Length}(\partial\Omega)}{2\pi} + o(\sqrt{\lambda})$ *as* $\lambda \to \infty$.

What's known:

- True when error term replaced with $O(\sqrt{\lambda})$ (Seeley, 1978)
- True when the set of periodic points of billiards has measure zero (whatever that means...) (Ivrii, 1980).

Inverse spectral problems

For a given drum $\Omega \subset \mathbb{R}^2$, the eigenvalues of $-\Delta$ (for zero boundary conditions) form a sequence

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Lots of questions to ask!!

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- For which sequences $\{\lambda_n\}_{n=1}^{\infty}$ can we find $\Omega \subset \mathbb{R}^2$ such that $\text{Spec}(\Omega) = \{\lambda_n\}_{n=1}^{\infty}$?
- Many other questions! Small deformations, Higher dimensions / Riemannian manifolds, other operators, etc.

Inverse spectral problems

CAN ONE HEAR THE SHAPE OF A DRUM?

MARK KAC, The Rockefeller University, New York

To George Eugene Uhlenbeck on the occasion of his sixty-fifth birthday

> "La Physique ne nous donne pas seulement l'occasion de résoudre des problèmes . . . , elle nous fait presentir la solution." H. POINCARÉ.

Inverse spectral problems

ONE CANNOT HEAR THE SHAPE OF A DRUM

CAROLYN GORDON, DAVID L. WEBB, AND SCOTT WOLPERT

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Fig. 10

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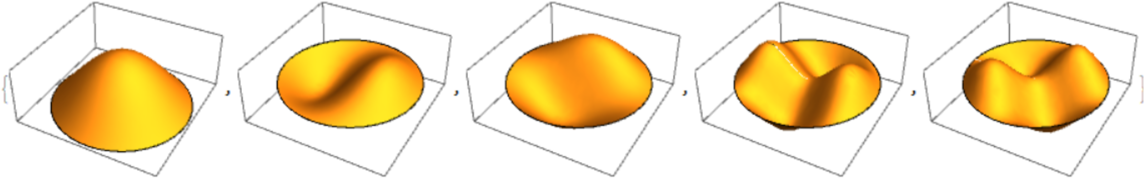

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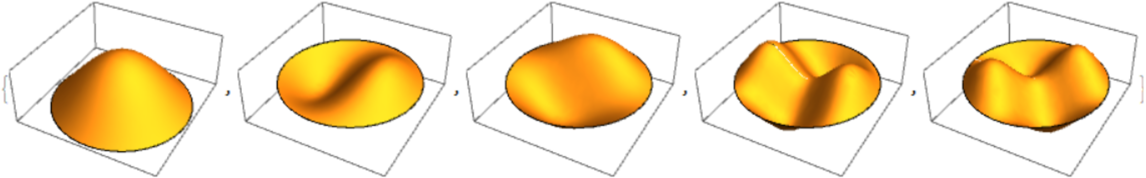

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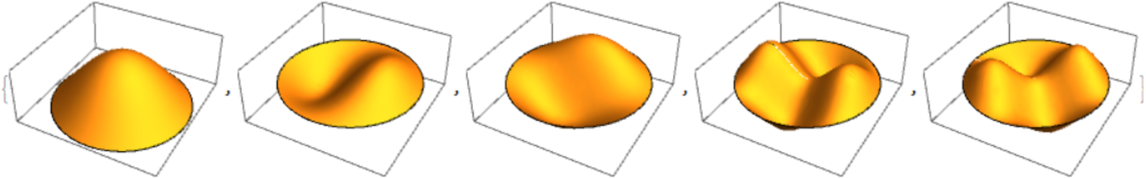

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Thus, $\operatorname{Spec}(\Omega) = \operatorname{Spec}(D) \implies \Omega$ has the same area and perimeter as a disk.Actually the only shape that can satisfy this condition is the disk itself (isoperimetric inequality). □

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- Ellipses with small eccentricity are spectrally determined (Hezari – Zelditch, 2019)
- The problem is still open for general ellispes...

More questions

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- Can we hear the shape of a drum by striking it at different points?
- Is the function Spec: $\{\Omega \subset \mathbb{R}^2\} \to \mathbb{R}_+^{\mathbb{N}}$ continuous (In a reasonably defined sense)?

Thank you for your attention!