

Fun with Diagram Algebras

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September 17, 2021

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- Original work on the representation theory: Martin (1991), Westbury (1995)
- Review paper by Ridout & Saint-Aubin (2014) discusses the representation theory very thoroughly

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My task:

- Adapt the Ridout & Saint-Aubin paper to study the three-parameter one-boundary Temperley-Lieb algebra $1\text{BTL}_n(\beta; \beta_1, \beta_2)$

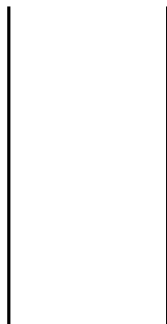
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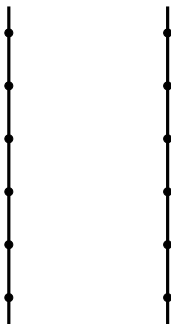
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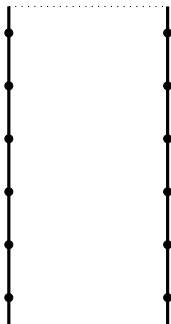
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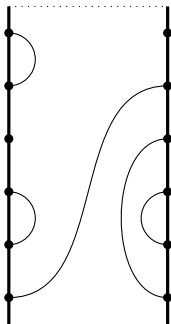
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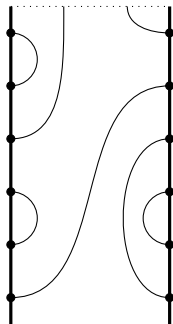
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- Strings may connect pairs of nodes



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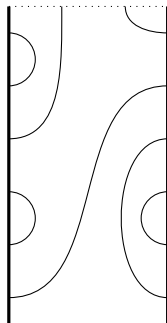
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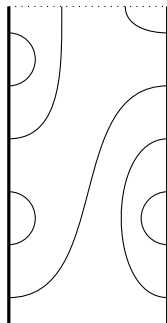
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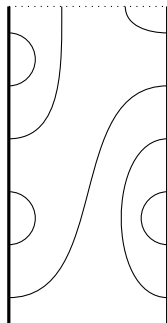
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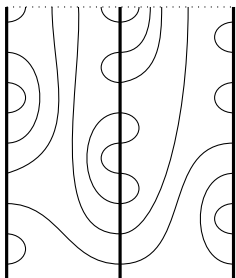
Introduce 1BTL_n as the complex vector space with the set of all n -diagrams as basis.

Diagram operations: multiplication in $1\text{BTL}_n(\beta; \beta_1, \beta_2)$

We now want to multiply n -diagrams.

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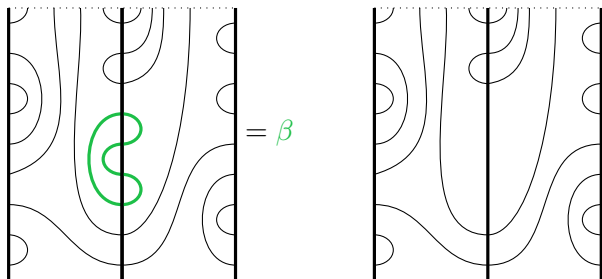
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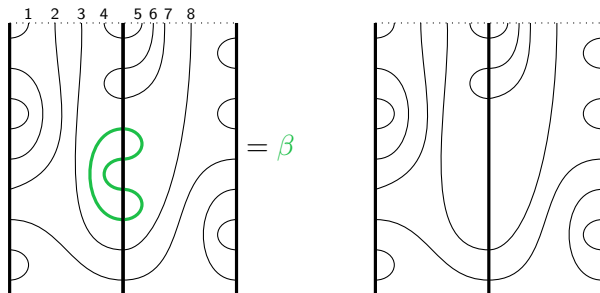
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- Draw the diagrams next to each other
- Replace each loop with a factor of β

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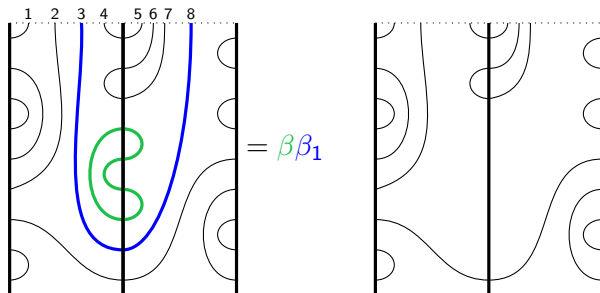
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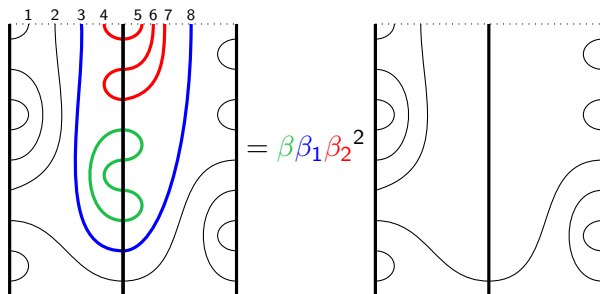
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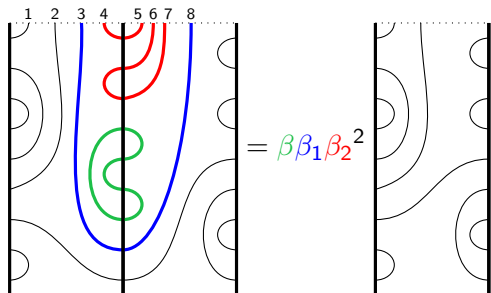
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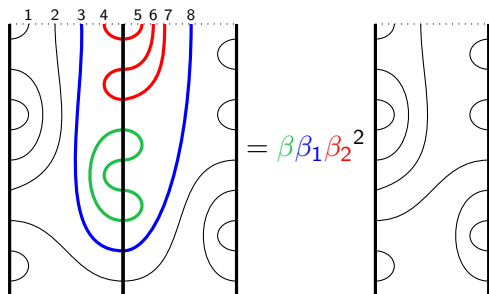
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- Replace each boundary arc with odd (even) left connection by a factor of β_1 (β_2)
- Remove the middle line and tighten the strings
- Extend this bilinearly to all of $1\text{BTL}_n(\beta; \beta_1, \beta_2)$

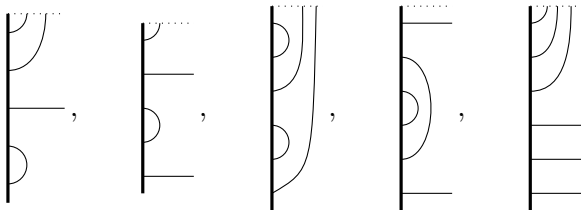
Introducing standard modules $\mathcal{V}_{n,d}$

We now want to construct a representation of $1\text{BTL}_n(\beta; \beta_1, \beta_2)$.

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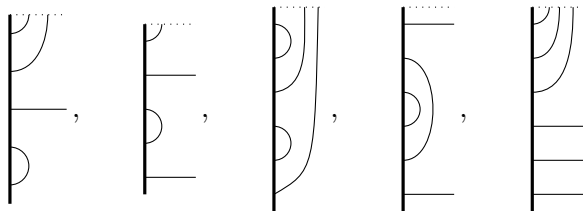
- Each n -diagram can be cut in half vertically to make *half-diagrams*, e.g.:



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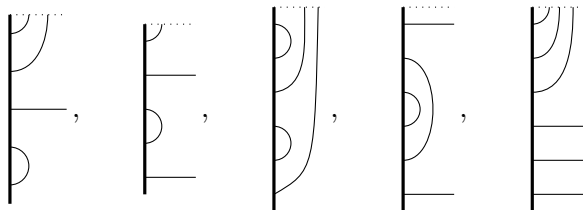


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 - a *defect*, sticking straight out to the right.
- No crossing of strings! Also, links and boundary links cannot pass over defects.

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We now want to define the action of $1\text{BTL}_n(\beta; \beta_1, \beta_2)$ on $\mathcal{V}_{n,d}$ to make it a *(left) $1\text{BTL}_n(\beta; \beta_1, \beta_2)$ -module*.

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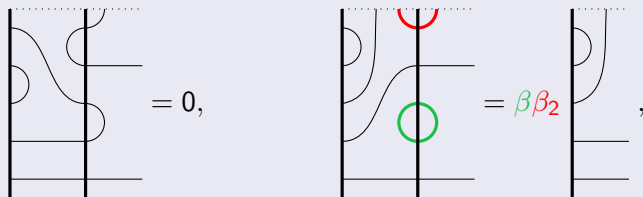
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- Define for diagrams and half-diagrams and extend bilinearly
- Draw side-by-side, replace loops and boundary arcs with factors of β , β_1 , β_2 as before, then tighten the strings
- Must preserve number of defects d ; if not, set the result to 0

Diagram operations: action on standard modules $\mathcal{V}_{n,d}$

Examples

In $\mathcal{V}_{5,2}$, we have



while in $\mathcal{V}_{5,1}$ we have

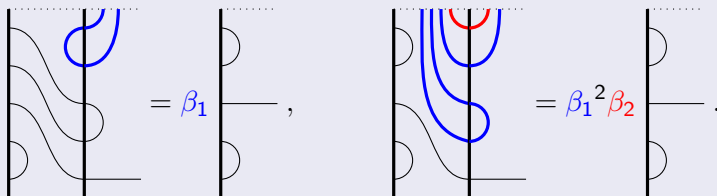


Diagram operations: outer product

From a pair of n -half-diagrams with the same number of defects d , we can construct an n -diagram.

Diagram operations: bilinear form on $\mathcal{V}_{n,d}$

What if we put two (n, d) -half-diagrams back-to-back instead?

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$$x = \text{diagram}, \quad y = \text{diagram}, \quad \langle x, y \rangle = \text{diagram} \in \mathbb{C}$$

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What if we put two (n, d) -half-diagrams back-to-back instead?

The diagrammatic equation shows three terms separated by commas. The first term is labeled $x =$ and consists of a vertical line with two horizontal lines at the bottom. A semi-circle on the left side of the line connects the two horizontal lines. A larger semi-circle on the right side of the line connects the top of the vertical line to the top of the lower semi-circle. The second term is labeled $y =$ and consists of a vertical line with two horizontal lines, one above and one below the vertical line. A semi-circle on the left side of the line connects the upper horizontal line to the vertical line. The third term is labeled $\langle x, y \rangle =$ and consists of a vertical line with two horizontal lines, one above and one below. A semi-circle on the left side of the line connects the upper horizontal line to the vertical line. A larger semi-circle on the right side of the line connects the top of the vertical line to the top of the lower semi-circle. The entire equation is followed by $\in \mathbb{C}$.

Extending this bilinearly to $\mathcal{V}_{n,d} \times \mathcal{V}_{n,d}$ would give a *bilinear form*:

- Like an inner product, except without some restrictions like $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Rightarrow x = 0$

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- Bilinear NOT sesquilinear

Diagram operations: bilinear form on $\mathcal{V}_{n,d}$

For the bilinear form to be useful, we would like to have

$$|x \ y|z \quad = \quad x \langle y, z \rangle$$

for all $x, y, z \in \mathcal{V}_{n,d}$, e.g.:

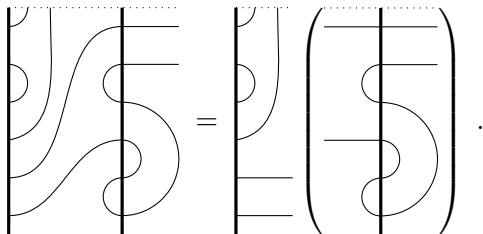
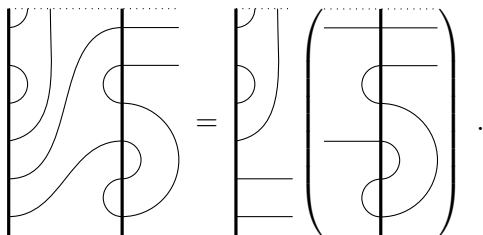


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Observing that $|x \ y| z \propto x$ for all $x, y, z \in \mathcal{V}_{n,d}$, this is possible, and uniquely defines a bilinear form $\langle \cdot, \cdot \rangle : \mathcal{V}_{n,d} \times \mathcal{V}_{n,d} \rightarrow \mathbb{C}$.

Diagram operations: bilinear form on $\mathcal{V}_{n,d}$

Examples

In $\mathcal{V}_{6,2}$, we have

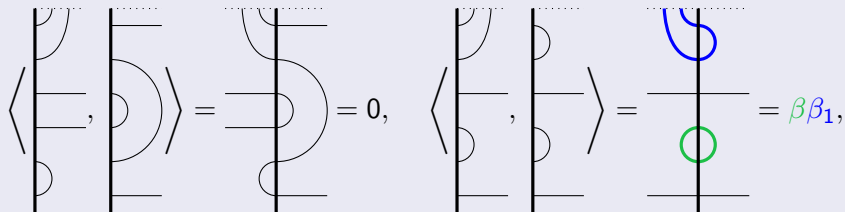
A diagrammatic equation in $\mathcal{V}_{6,2}$. On the left, two diagrams are multiplied. The first diagram has a vertical line with a cap at the top, a cup at the bottom, and a crossing. The second diagram has a vertical line with a cap at the top, a cup at the bottom, and a crossing. The product is shown as a single diagram with a vertical line and a crossing, which is equal to 0.

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Diagram operations: bilinear form on $\mathcal{V}_{n,d}$

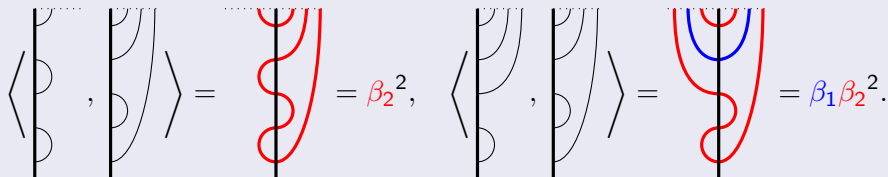
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Diagrammatic equations in $\mathcal{V}_{6,2}$. The first equation shows a product of two diagrams (left and right) equal to a single diagram with a loop, which is equal to 0. The second equation shows a product of two diagrams equal to a diagram with a blue loop and a green circle, which is equal to $\beta\beta_1$.

while in $\mathcal{V}_{5,0}$ we have



Diagrammatic equations in $\mathcal{V}_{5,0}$. The first equation shows a product of two diagrams equal to a diagram with a red loop, which is equal to β_2^2 . The second equation shows a product of two diagrams equal to a diagram with a blue loop and a red loop, which is equal to $\beta_1\beta_2^2$.

Bilinear form: Gram Matrices

Each of these bilinear forms has an associated *Gram matrix*.

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$$G_{3,1} = \begin{pmatrix} \begin{array}{c} \text{---} \\ \text{U} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ \text{U} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ \text{U} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \\ \begin{array}{c} \text{---} \\ \text{U} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ \text{O} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ \text{U} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \\ \begin{array}{c} \text{---} \\ \text{U} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ \text{U} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ \text{O} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \end{pmatrix} = \begin{pmatrix} \beta_1\beta_2 & \beta_1 & 0 \\ \beta_1 & \beta & 1 \\ 0 & 1 & \beta \end{pmatrix}$$

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$$G_{3,1} = \begin{pmatrix} \text{Diagram 1} & \text{Diagram 2} & \text{Diagram 3} \\ \text{Diagram 4} & \text{Diagram 5} & \text{Diagram 6} \\ \text{Diagram 7} & \text{Diagram 8} & \text{Diagram 9} \end{pmatrix} = \begin{pmatrix} \beta_1\beta_2 & \beta_1 & 0 \\ \beta_1 & \beta & 1 \\ 0 & 1 & \beta \end{pmatrix}$$

The diagrams in the matrix are:

- Row 1: A vertical line with a U-shaped arc above it; a vertical line with a loop above it; a vertical line with a loop and a horizontal line to the right above it.
- Row 2: A vertical line with a loop above it; a vertical line with a circle above it; a vertical line with a loop and a horizontal line to the right above it.
- Row 3: A vertical line with a loop and a horizontal line to the right above it; a vertical line with a loop and a horizontal line to the right above it; a vertical line with a circle above it.

Note that $\det(G_{n,d})$ must be a polynomial in β, β_1, β_2 .

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Note that $\det(G_{n,d})$ must be a polynomial in β, β_1, β_2 .

Keep this in mind! Very important later.

Structure of $\mathcal{V}_{n,d}$

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- If you start with a vector $v \in \mathcal{V}_{n,d}$, which other elements of $\mathcal{V}_{n,d}$ can you get to by acting on v with elements of the algebra $1\text{BTL}_n(\beta; \beta_1, \beta_2)$?

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- **Are there any subspaces you can get stuck in?**

Definition

A *submodule* of an A -module V , for an algebra A , is a subspace W of V such that

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Structure of $\mathcal{V}_{n,d}$

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An A -module V is called *irreducible* if it has no submodules other than $\{0\}$ and V itself.

Structure of $\mathcal{V}_{n,d}$

Question: Is $\mathcal{V}_{n,d}$ irreducible?

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Hence the technical goals of my thesis were:

- Find $\det(G_{n,d})$,
- Find when $\det(G_{n,d}) = 0$.

Determinant of the Gram matrix

Theorem

For any $\beta, \beta_1, \beta_2 \in \mathbb{C}$, the determinant of the Gram matrix $G_{n,d}$ is given by

$$\det(G_{n,d}) =$$

$$\left\{ \begin{array}{l} (-\beta_1)^{\binom{\frac{n-d}{2}-1}{} } \prod_{j=1}^{\frac{n-d}{2}} \left(\beta_1 U_{d+j-1} \left(\frac{\beta}{2} \right) - \beta_2 U_{d+j} \left(\frac{\beta}{2} \right) \right)^{\binom{\frac{n-d}{2}-j}{} } \\ \quad \times \prod_{k=1}^{\frac{n-d}{2}-1} \left(\beta_2 U_{k-1} \left(\frac{\beta}{2} \right) - \beta_1 U_k \left(\frac{\beta}{2} \right) \right)^{\binom{\frac{n-d}{2}-k-1}{} }, \quad d \equiv n \pmod{2}, \\ \\ \beta_2^{\binom{\frac{n-d-1}{2}}{} } \prod_{j=1}^{\frac{n-d-1}{2}} \left(\beta_2 U_{d+j-1} \left(\frac{\beta}{2} \right) - \beta_1 U_{d+j} \left(\frac{\beta}{2} \right) \right)^{\binom{\frac{n-d-1}{2}-j}{} } \\ \quad \times \prod_{k=1}^{\frac{n-d-1}{2}} \left(\beta_1 U_{k-1} \left(\frac{\beta}{2} \right) - \beta_2 U_k \left(\frac{\beta}{2} \right) \right)^{\binom{\frac{n-d-1}{2}-k}{} }, \quad d \not\equiv n \pmod{2}, \end{array} \right.$$

where U_m is the m th Chebyshev polynomial of the second kind.

When $\det(G_{n,d}) = 0$

Theorem

We have $\det(G_{n,d}) = 0$ if and only if $d < n$, and

- $\beta_{n,d} = 0$; or
- $\beta_{n,d} \neq 0$, $q \neq \pm 1$, and $\beta'_{n,d} \notin \{q\beta_{n,d}, q^{-1}\beta_{n,d}\}$, and
 - $\xi_{n,d} - (d+j+1)\lambda \in \pi\mathbb{Z}$ for some $j \in \mathbb{Z}$ with $1 \leq j \leq \lfloor \frac{n-d}{2} \rfloor$, or
 - $\xi_{n,d} + k\lambda \in \pi\mathbb{Z}$ for some $k \in \mathbb{Z}$ with $1 \leq k \leq \lfloor \frac{n-d-1}{2} \rfloor$; or
- $\beta_{n,d} \neq 0$, $q = \pm 1$, and
 - $\beta'_{n,d} = \frac{d+j}{d+j+1} q^{-1} \beta_{n,d}$ for some $j \in \mathbb{Z}$ with $1 \leq j \leq \lfloor \frac{n-d}{2} \rfloor$, or
 - $\beta'_{n,d} = \frac{k+1}{k} q \beta_{n,d}$ for some $k \in \mathbb{Z}$ with $1 \leq k \leq \lfloor \frac{n-d-1}{2} \rfloor$,

where $\beta = q + q^{-1}$, $q = e^{i\lambda}$, and $\xi_{n,d}$ comes from the parametrisation

$$\beta'_{n,d} = \frac{q - q^{-1} e^{2i\xi_{n,d}}}{1 - e^{2i\xi_{n,d}}} \beta_{n,d},$$

where $\beta_{n,d} = \beta_1$ and $\beta'_{n,d} = \beta_2$ if $d \equiv n \pmod{2}$, or vice versa if $d \not\equiv n \pmod{2}$.

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- Checked the $G_{n,d} = 0$ cases, so found precisely when each $\mathcal{V}_{n,d}$ is irreducible
- Can deduce from this (and further arguments from Graham & Lehrer (1996)) the values of β , β_1 , β_2 for which $1\text{BTL}_n(\beta; \beta_1, \beta_2)$ is *semisimple*, meaning any finite-dimensional $1\text{BTL}_n(\beta; \beta_1, \beta_2)$ -module is isomorphic to a direct sum of irreducible modules

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 - Found that $1\text{BTL}_n(\beta; \beta_1, \beta_2)$ is semisimple for generic parameter values, but not all parameter values
 - $1\text{BTL}_n(\beta; \beta_1, \beta_2)$ is semisimple if and only if $\det(G_{n,d}) \neq 0$ for all $0 \leq d \leq n$.

Where to from here?

- Consider *indecomposable* $1\text{BTL}_n(\beta; \beta_1, \beta_2)$ -modules, i.e. those which cannot be expressed as a direct sum of two nonzero submodules

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- Apply similar procedures to other algebras e.g. two-boundary and periodic Temperley-Lieb algebras, BMW algebra (crossing of strings allowed)
- Connect back to physics – look at the Hamiltonians for the corresponding lattice models, and their energy eigenvalues

Determinant bloopers: Sierpinski triangle

