Fun with Diagram Algebras

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The University of Queensland

September 17, 2021

Context

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- Original work on the representation theory: Martin (1991), Westbury (1995)
- Review paper by Ridout & Saint-Aubin (2014) discusses the representation theory very thoroughly

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My task:

- Adapt the Ridout & Saint-Aubin paper to study the three-parameter one-boundary Temperley-Lieb algebra $1\mathrm{BTL}_n(\beta; \beta_1, \beta_2)$

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We now introduce *n*-diagrams.

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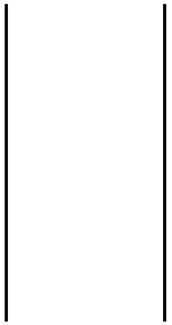

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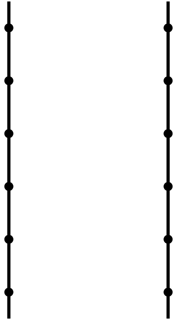

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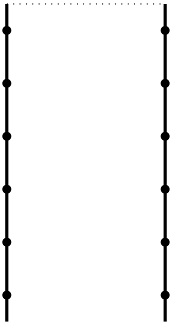

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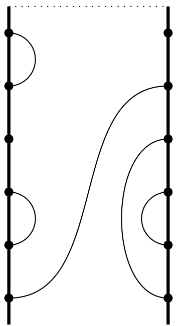

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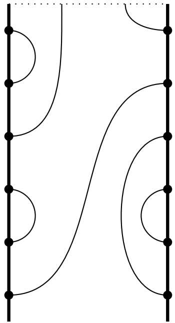

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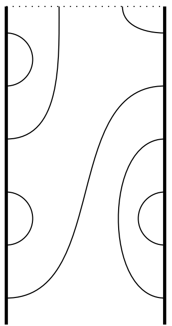

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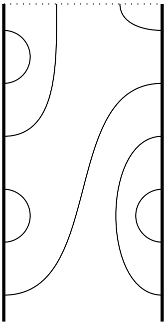

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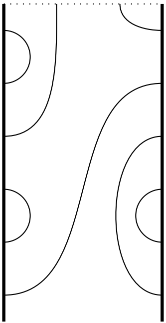

Introduce $1\mathrm{BTL}_n$ as the complex vector space with the set of all n-diagrams as basis.

Diagram operations: multiplication in $1\mathrm{BTL}_n(\beta; \beta_1, \beta_2)$

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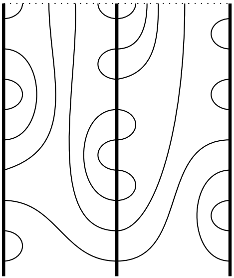

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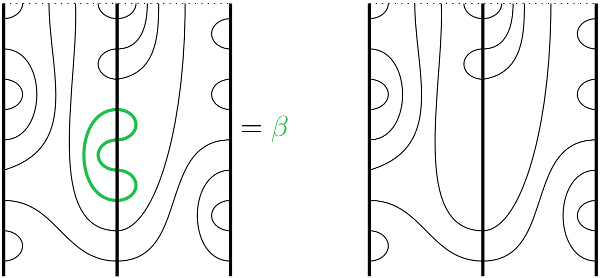

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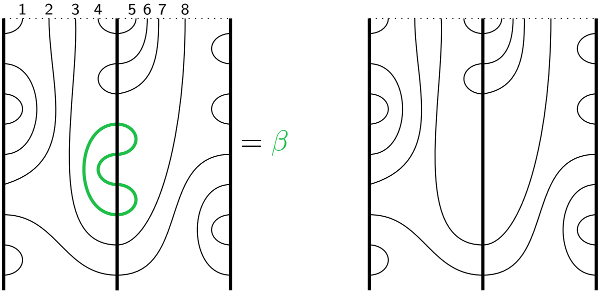

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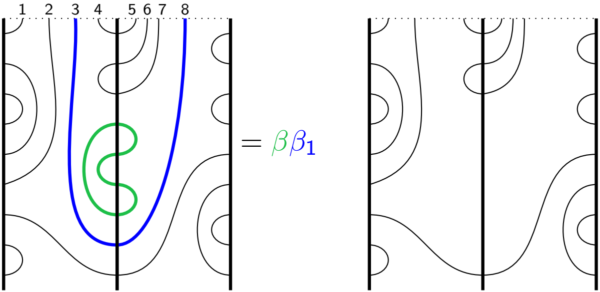

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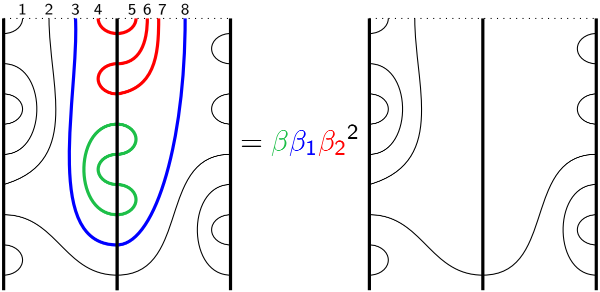

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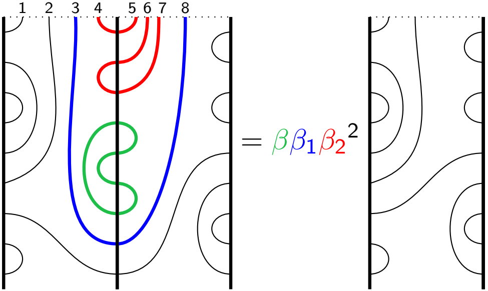

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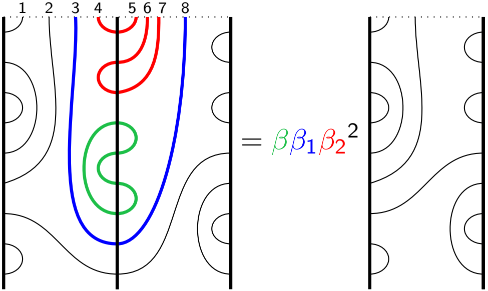

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- Extend this bilinearly to all of $1\mathrm{BTL}_n(\beta; \beta_1, \beta_2)$

Introducing standard modules $\mathcal{V}_{n,d}$

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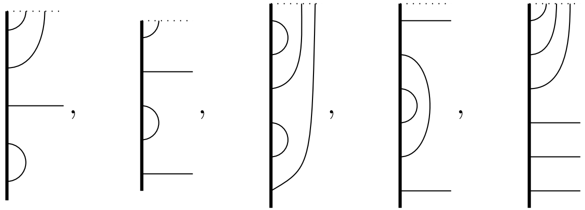

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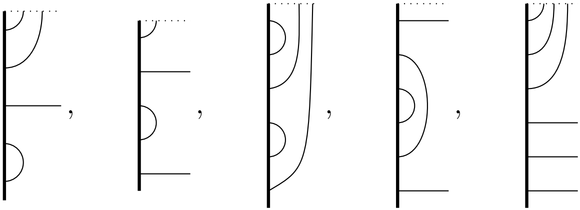

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  - a *link*, connecting it to another node,
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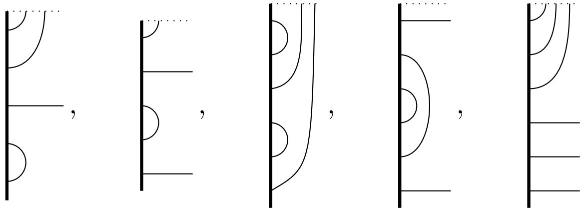

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    - a *defect*, sticking straight out to the right.
- No crossing of strings! Also, links and boundary links cannot pass over defects.

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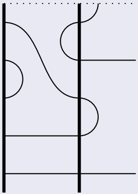
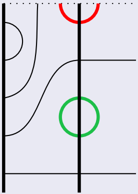
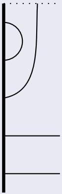
- Define for diagrams and half-diagrams and extend bilinearly
- Draw side-by-side, replace loops and boundary arcs with factors of β, β_1, β_2 as before, then tighten the strings
- Must preserve number of defects d; if not, set the result to 0

Diagram operations: action on standard modules $\mathcal{V}_{n,d}$

Examples

In $\mathcal{V}_{5,2}$, we have

 $= 0,$  $= \beta\beta_2$  ,

while in $\mathcal{V}_{5,1}$ we have

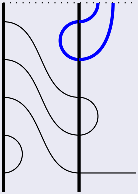
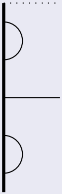
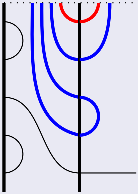
 $= \beta_1$  ,  $= {\beta_1}^2\beta_2$  .

Diagram operations: outer product

From a pair of n-half-diagrams with the same number of defects d, we can construct an n-diagram.

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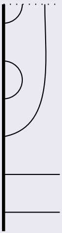
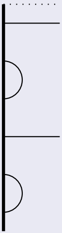
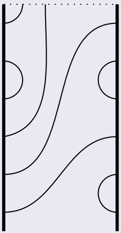
$x =$  , $y =$  , $|x \;\; y| :=$  $\in 1\mathrm{BTL}_6$.

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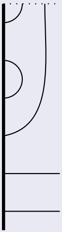
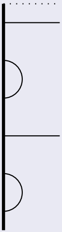
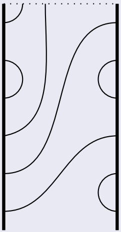
$x =$  , $y =$  , $|x \;\; y| :=$  $\in \mathrm{1BTL}_6$.

This can be extended bilinearly to a map $|\cdot \;\; \cdot| : \mathcal{V}_{n,d} \times \mathcal{V}_{n,d} \to \mathrm{1BTL}_n$.

Diagram operations: bilinear form on $\mathcal{V}_{n,d}$

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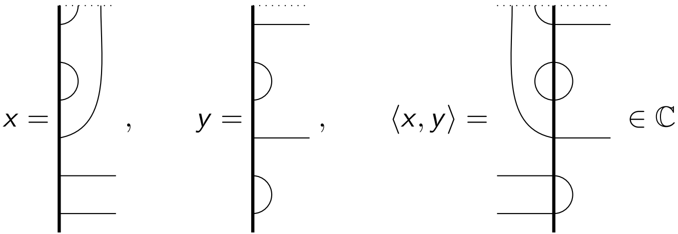

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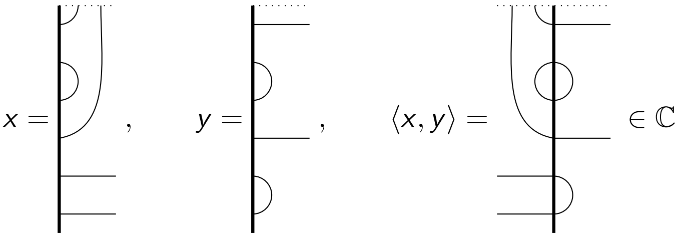

Extending this bilinearly to $\mathcal{V}_{n,d} \times \mathcal{V}_{n,d}$ would give a *bilinear form*:

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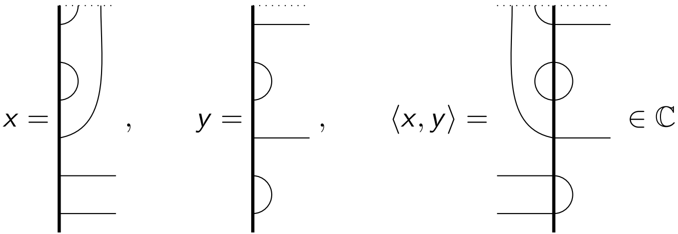

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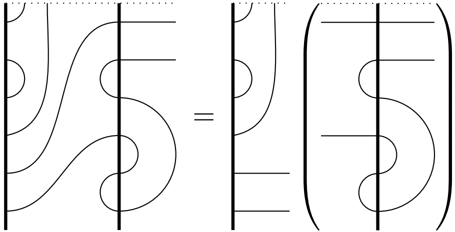
- Bilinear NOT sesquilinear

Diagram operations: bilinear form on $\mathcal{V}_{n,d}$

For the bilinear form to be useful, we would like to have

$$|x \quad y|\, z \qquad = \qquad x\,\langle y, z\rangle$$

for all $x, y, z \in \mathcal{V}_{n,d}$, e.g.:



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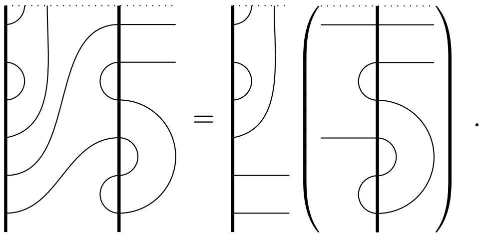

Observing that $|x \quad y|\, z \propto x$ for all $x, y, z \in \mathcal{V}_{n,d}$, this is possible, and uniquely defines a bilinear form $\langle \cdot, \cdot \rangle : \mathcal{V}_{n,d} \times \mathcal{V}_{n,d} \to \mathbb{C}$.

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Examples

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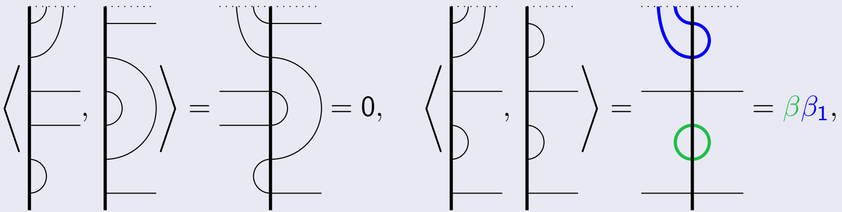

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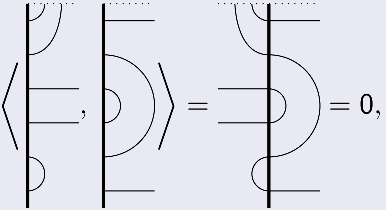

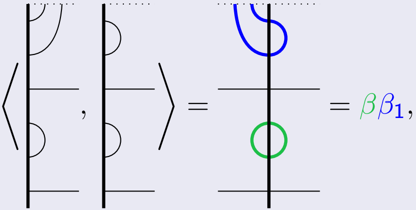

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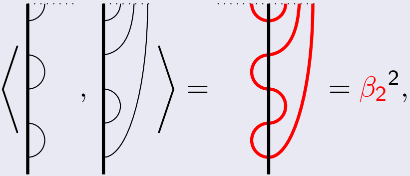

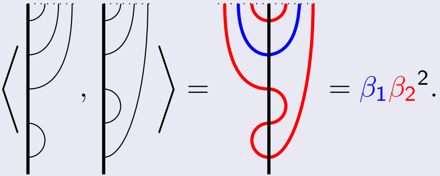

Bilinear form: Gram Matrices

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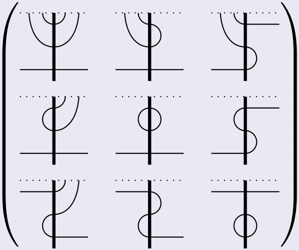

$$= \begin{pmatrix} \beta_1\beta_2 & \beta_1 & 0 \\ \beta_1 & \beta & 1 \\ 0 & 1 & \beta \end{pmatrix}$$

Bilinear form: Gram Matrices

Each of these bilinear forms has an associated *Gram matrix*.

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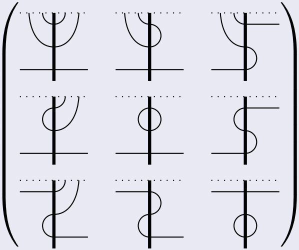

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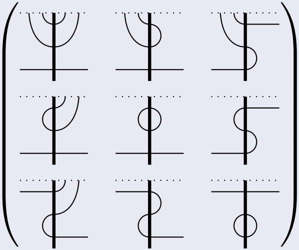

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Note that $\det(G_{n,d})$ must be a polynomial in β, β_1, β_2.

Keep this in mind! Very important later.

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- If you start with a vector $v \in \mathcal{V}_{n,d}$, which other elements of $\mathcal{V}_{n,d}$ can you get to by acting on v with elements of the algebra $\mathrm{1BTL}_n(\beta; \beta_1, \beta_2)$?

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- **Are there any subspaces you can get stuck in?**

Structure of $\mathcal{V}_{n,d}$

Definition

A *submodule* of an A-module V, for an algebra A, is a subspace W of V such that

$$\forall a \in A,\ \forall w \in W, \qquad aw \in W.$$

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An A-module V is called *irreducible* if it has no submodules other than $\{0\}$ and V itself.

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Hence the technical goals of my thesis were:

- Find $\det(G_{n,d})$,
- Find when $\det(G_{n,d}) = 0$.

Determinant of the Gram matrix

Theorem

For any $\beta, \beta_1, \beta_2 \in \mathbb{C}$, the determinant of the Gram matrix $G_{n,d}$ is given by

$$\det(G_{n,d}) = \begin{cases} (-\beta_1)^{\left(\frac{n}{\frac{n-d}{2}}-1\right)} \prod_{j=1}^{\frac{n-d}{2}} \left(\beta_1 U_{d+j-1}\left(\frac{\beta}{2}\right) - \beta_2 U_{d+j}\left(\frac{\beta}{2}\right)\right)^{\left(\frac{n}{\frac{n-d}{2}-j}\right)} \\ \quad \times \prod_{k=1}^{\frac{n-d}{2}-1} \left(\beta_2 U_{k-1}\left(\frac{\beta}{2}\right) - \beta_1 U_k\left(\frac{\beta}{2}\right)\right)^{\left(\frac{n}{\frac{n-d}{2}-k-1}\right)}, & d \equiv n \mod 2, \\ \beta_2^{\left(\frac{n}{\frac{n-d-1}{2}}\right)} \prod_{j=1}^{\frac{n-d-1}{2}} \left(\beta_2 U_{d+j-1}\left(\frac{\beta}{2}\right) - \beta_1 U_{d+j}\left(\frac{\beta}{2}\right)\right)^{\left(\frac{n}{\frac{n-d-1}{2}-j}\right)} \\ \quad \times \prod_{k=1}^{\frac{n-d-1}{2}} \left(\beta_1 U_{k-1}\left(\frac{\beta}{2}\right) - \beta_2 U_k\left(\frac{\beta}{2}\right)\right)^{\left(\frac{n}{\frac{n-d-1}{2}-k}\right)}, & d \not\equiv n \mod 2, \end{cases}$$

where U_m is the mth Chebyshev polynomial of the second kind.

When $\det(G_{n,d}) = 0$

Theorem

We have $\det(G_{n,d}) = 0$ *if and only if* $d < n$, *and*

- $\beta_{n,d} = 0$; *or*
- $\beta_{n,d} \neq 0$, $q \neq \pm 1$, *and* $\beta'_{n,d} \notin \{q\beta_{n,d}, q^{-1}\beta_{n,d}\}$, *and*
  - $\xi_{n,d} - (d+j+1)\lambda \in \pi\mathbb{Z}$ *for some* $j \in \mathbb{Z}$ *with* $1 \leq j \leq \left\lfloor \frac{n-d}{2} \right\rfloor$, *or*
  - $\xi_{n,d} + k\lambda \in \pi\mathbb{Z}$ *for some* $k \in \mathbb{Z}$ *with* $1 \leq k \leq \left\lfloor \frac{n-d-1}{2} \right\rfloor$; *or*
- $\beta_{n,d} \neq 0$, $q = \pm 1$, *and*
  - $\beta'_{n,d} = \frac{d+j}{d+j+1} q^{-1} \beta_{n,d}$ *for some* $j \in \mathbb{Z}$ *with* $1 \leq j \leq \left\lfloor \frac{n-d}{2} \right\rfloor$, *or*
  - $\beta'_{n,d} = \frac{k+1}{k} q \beta_{n,d}$ *for some* $k \in \mathbb{Z}$ *with* $1 \leq k \leq \left\lfloor \frac{n-d-1}{2} \right\rfloor$,

where $\beta = q + q^{-1}$, $q = e^{i\lambda}$, *and* $\xi_{n,d}$ *comes from the parametrisation*

$$\beta'_{n,d} = \frac{q - q^{-1} e^{2i\xi_{n,d}}}{1 - e^{2i\xi_{n,d}}} \beta_{n,d},$$

where $\beta_{n,d} = \beta_1$ *and* $\beta'_{n,d} = \beta_2$ *if* $d \equiv n \mod 2$, *or vice versa if* $d \not\equiv n \mod 2$.

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- Can deduce from this (and further arguments from Graham & Lehrer (1996)) the values of β, β_1, β_2 for which $1\mathrm{BTL}_n(\beta;\beta_1,\beta_2)$ is *semisimple*, meaning any finite-dimensional $1\mathrm{BTL}_n(\beta;\beta_1,\beta_2)$-module is isomorphic to a direct sum of irreducible modules

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  - Found that $1\mathrm{BTL}_n(\beta;\beta_1,\beta_2)$ is semisimple for generic parameter values, but not all parameter values
  - $1\mathrm{BTL}_n(\beta;\beta_1,\beta_2)$ is semisimple if and only if $\det(G_{n,d}) \neq 0$ for all $0 \leq d \leq n$.

Where to from here?

- Consider *indecomposable* $1\mathrm{BTL}_n(\beta;\beta_1,\beta_2)$-modules, i.e. those which cannot be expressed as a direct sum of two nonzero submodules

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- Apply similar procedures to other algebras e.g. two-boundary and periodic Temperley-Lieb algebras, BMW algebra (crossing of strings allowed)
- Connect back to physics – look at the Hamiltonians for the corresponding lattice models, and their energy eigenvalues

Determinant bloopers: Sierpinski triangle

1 1
-1 1 1
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